Real Number System

The set of Natural Number denoted by N is defined as $N = \{1, 2, 3, \ldots\}$

The set of whole numbers denoted by ω is defined as $\omega = \{0, 1, 2, \ldots\}$.

The set of Integers denoted by Z is defined as $Z = \{\ldots -3, -2, -1, 0, 1, 2, 3 \ldots\}$

The set of Rational Numbers denoted by Q is defined as $Q = \{P/q \,/\, P, q \in Z,\ q \neq 0\}$

The set of Irrational Numbers denoted by I i[s] defined as the numbers which are non-repeati[ng] and non-terminating numbers.

Ex: $\sqrt{2} = 1.444\ldots$

The set of Real numbers denoted by $\mathbb{R}$ is the union of set of rational Numbers and set [of] irrational numbers i.e.,

$$\mathbb{R} = Q \cup I = Q \cup (\mathbb{R} - Q)$$

The Real Numbers are represented on a rea[l] line. The real numbers are also denoted as $(-\infty$

Some properties of real numbers:

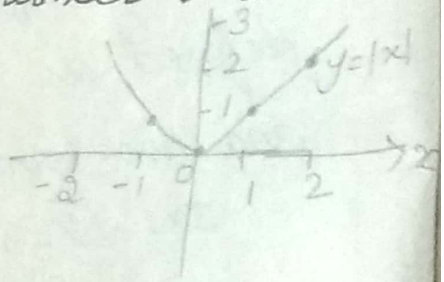
1. * The set of real numbers satisfies order axioms

The order axioms (that can be taken as it is true without any proof) are

(i) For $a, b \in \mathbb{R}$ one and only one of $a > b$, $a = b$ or $a < b$ is true (Trichotomy)

* For $a, b, c \in \mathbb{R}$, $a > b$, $b > c \Rightarrow a > c$ (Transitivites)
* For $a, b, c \in \mathbb{R}$, $a > b \Rightarrow a + c > b + c$ (monotone property)
* For $a, b, c \in \mathbb{R}$, $a > b$, $c > 0 \Rightarrow ac > bc$

2. **Absolute (or) modules** of a real number x is **defined** as $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$



3. Triangle Inequality:-

For any $x, y \in \mathbb{R}$, $|x+y| \leq |x| + |y|$

Finite and Infinite subsets of $\mathbb{R}$:-

A nonempty subset of S of $\mathbb{R}$ is said to finite, if there exist a bijective function $f : S \to \{1, 2, \ldots n\}$ for some $n \in N$, if S is not finite then it is said to be infinite set.

Aggregate: A nonempty subset of $\mathbb{R}$ is called an aggregate.

Boundedness of subsets of $\mathbb{R}$:

Bounded above: An aggregate S is said to be bounded above, if there exist $K_1 \in R$ such that $x \in S \Rightarrow x \le K_1$. K_1 is called an upperbound of S.

Ex: $Z = \{ ----3, -2, -1 \}$ is bounded above.

$\because x \le -1 \; \forall \; x \in Z^{-1}$, -1 is a ~~lower~~ upper bound of Z

Least upper bound (or) Supremum:-

An upper bound u of an aggregate 'S' is said to be the least upper bound (or) supremum If any real number less than 'u' is not an upper bond of 'S'.

Ex: For Z^-, -1 is the least upper bound.

Bounded below: An aggregate S is said to be bounded below, if $\exists \; K_2 \in R$ such that $x \in S \Rightarrow x \ge K_2$ (or) $K_2 \le x$; K_2 is called a lower bound of S.

Ex: $Z^+ = \{1, 2, 3 - - \}$ is bounded below.

$\because 1 \le x \; \forall \; x \in Z^+$, 1 is a lower bound of Z^+

Greatest lower bound (or) infimum:

An lower bound u of an aggregate 'S' is said to be the greatest lower band or infimum if any real number greater that 'u' is not an lower bound of 'S'.

Ex: For, Z^+, 1 is the greatest lower bound

Bounded Set: An aggregate 'S' is said to be bounded set if it is both bounded above and bounded below.

Ex: $S = \{1/x \ / x \in N\} = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$

Here, $0 < \frac{1}{x} < 1, \ \forall \ x \in N$

$\therefore$ 0 is lower bound & 1 is an upper bound of S

$\therefore$ 'S' is a bounded set.

Note:

1. If an aggregate S is bounded above & supremum Exists then it is unique.

2. If an aggregate S is bounded below & infimum exists then it unique.

3. If u is an upper bound of S & u belogs to 'S' then supremum S=u [$u \in S$ then $\sup S = u$]

4. If ~~u~~ v is a lower bound of S ~~of~~ & $v \in S$ then $\text{Inf } S = v$

The completness Axiom:

Every non-empty set of real numbers which bounded above has supremum.

From the above theorem we can prove any non-empty set of real number which is bounded below has Infimum

Archemedian Property:

If $x, y \in \mathbb{R}$ and $x > 0$ then $\exists\ n \in Z^+ \ni nx > y$

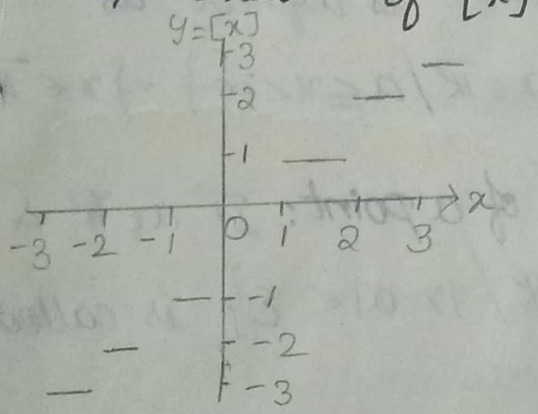
Integral part of a real number:

If x is a real number then the integral part of x denoted by $[x]$ is defined as the greatest integer which is less than x

(or)

$[x] = n$ if $n \le x < n+1$

Graphical representation of $[x]$



Note:

1. Between any two different real numbers there is a rational number.

2. Between any two different real numbers there is

irrational number.

Intervals: Let $a, b \in \mathbb{R}$

(1) **Closed Interval**: The closed interval denoted by $[a, b]$ is defined as $[a,b] = \{x \in \mathbb{R} / a \leq x \leq b\}$

(2) **Open interval**: Let $a, b \in \mathbb{R}$ the open interval $(a,b) = \{x \in \mathbb{R} / a < x < b\}$

(3) **Semi closed or semi open intervals**: For $a, b \in \mathbb{R}$, semi closed & semi open intervals are defined as $[a, b) = \{x \in \mathbb{R} / a \leq x < b\}$, $(a, b] = \{x \in \mathbb{R} / a < x \leq b\}$

(4) **Infinite open interval**: For $a \in \mathbb{R}$ the infinite open interval is defined as $(a, \infty) = \{x \in \mathbb{R} / a < x < \infty\}$

$= \{x \in \mathbb{R} / x > a\}$

(5) **Infinite semiclosed interval**: For $a \in \mathbb{R}$, the infinite semiclosed interval is defined as

$$[a, \infty) = \{x \in \mathbb{R} / a \leq x < \infty\} = \{x \in \mathbb{R} / x \geq a\}$$

Neighbourhood of a point: If $a \in \mathbb{R}$ & $\varepsilon > 0$ then the set $\{x \in \mathbb{R} / |x-a| < \varepsilon\}$ is called ε-neighbourhood of a.

It is denoted by $N_\varepsilon(a)$.

$$N_\varepsilon(a) = \{x \in \mathbb{R} / |x-a| < \varepsilon\}$$
$$= \{x \in \mathbb{R} / -\varepsilon < x-a < \varepsilon\}$$
$$= \{x \in \mathbb{R} / a-\varepsilon < x < a+\varepsilon\}$$
$$= (a-\varepsilon, a+\varepsilon)$$

Ex: $\frac{1}{4}$ neighbourhood of $2 \in \mathbb{R}$ is

$$N_{\frac{1}{4}}(2) = \{x \in \mathbb{R} / 2 - \tfrac{1}{4} < x < 2 + \tfrac{1}{4}\}$$
$$= \{x \in \mathbb{R} / \tfrac{7}{4} < x < \tfrac{9}{4}\}$$
$$= \left(\tfrac{7}{4}, \tfrac{9}{4}\right)$$

Deleted Neighbourhood of $a \in \mathbb{R}$:-

Deleted Neighbourhood of $a \in \mathbb{R}$ is defined as
$\{x \in \mathbb{R} / |x-a| < \varepsilon, x \neq a\} = (a-\varepsilon, a) \cup (a, a+\varepsilon)$

Definition:

An agreegate S is said to be neighbourhood of $a \in \mathbb{R}$, if $\exists\, \varepsilon > 0 \ni (a-\varepsilon, a+\varepsilon) \subseteq S$.

Ex: An (a,b) is a neighbourhood of all its points.
$[a,b]$ is a neighbourhood of each of its points except a & b.

Limit point of Subset of $\mathbb{R}$:

Def 1: A point $P \in \mathbb{R}$ is said to be a limit point of an aggregate S if every neighbourhood P has a point of S other than P itself.

Def 2: A point $P \in \mathbb{R}$ is said to be a limit point of an aggregate S if every neighbourhood of P has a infinite number of points of S.

Ex: The set of integers Z has no limit points.

Sol: For $P \in \mathbb{R}$ $\exists$ unique integer $n \in Z \ni n \leq P < n+1$
If we take
$\varepsilon = \min\{P-n, n+1-P\}$

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then $(p-\varepsilon, p+\varepsilon)$ does not contain any integer other than p.

$\therefore$ p is a not limit point of Z.

For $p = 1.7$

$1 < 1.7 < 2$

$\varepsilon = \min\{1.7-1, 2-1.7\}$

$= \min\{0.7, 0.3\}$

$= 0.3$

$(p-\varepsilon, p+\varepsilon)$

$= (1.7-0.3, 1.7+0.3)$

$= (1.4, 2)$ has no integer

$\therefore$ $p = 1.7$ is not a limit in A.

Derived set: The set of all limit points of an aggregate S is called the derived set of S. and is denoted by $D(S)$.

Ex: $D(Z) = \phi$

$D(\phi) = R$

$D(R) = R$

Interior point: A point $P \in S$ is said to be an interior point of S. If there exist at least one neighbourhood which is entirely containing S.

Ex: Every point (a,b) is an interior point of (a,b)

Every point $[a,b]$ is an interior point except a and b.

open set: If every point of an aggregate S is an interior point then S is called an open set.

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Ex: (a,b) is an openset

$\mathbb{R}, \phi$ are opensets

Closed set: The set S is said to be a closed set if its compliment is an open set.

i.e., $\mathbb{R}-S$ is an open set.

Ex: $\because \mathbb{R}-\phi=\mathbb{R}$ is open,

ϕ is closed

$\because \mathbb{R}-\mathbb{R}=\phi$ is open, $\mathbb{R}$ is closed.

Countable and uncountable sets:

A set S is said to be countable if

(i) $S=\phi$ (ii) S is ~~i~~finite (or) (iii) S is ennumarable.

i.e., there is a bijection between S and μ

If S is not countable then it is said to be an uncountable set.

Ex: Set of even integers is a countable set.

Set of real number is not a countable.

Sequences

Def: A function $S: Z^+ \to \mathbb{R}$ is called a real sequ[ence] or sequence.

The terms of the sequence $S(1), S(2), S(3) \ldots$ [are] denoted as $S_1, S_2, S_3 \ldots$ respectively.

A sequence is denoted by $\{S_n\}$

Ex: $\{S_n\} = \{n^2\}$ is a sequence, the terms in the sequence are $S_1 = 1^2 = 1$, $S_2 = 2^2 = 4$, $S_3 = 3^2 = 9 \ldots$

Methods of defining sequence:

1. Defining a sequence by one or more formulae so that the n^{th} term for each $n \in N$ can be found

Ex: $S_n = \begin{cases} 1/n & \text{if 'n' is even} \\ -1/n & \text{if 'n' is odd} \end{cases}$. The terms in this seq[uence] are $S_1 = -1$, $S_2 = 1/2$, $S_3 = -1/3$, $S_4 = 1/4$ ---- $S_n = \frac{(-1)^n}{n}$

2. Defining a sequence by a recursion formula i.e, expressing n^{th} term in terms of $(n-1)^{th}$ term

Ex: (i) $S_1 = \sqrt{2}$, $S_{n+1} = \sqrt{2 + S_n}$, $n = 1, 2, \ldots$ Terms in this sequence are $S_1 = \sqrt{2}$, $S_2 = \sqrt{2 + S_1} = \sqrt{2 + \sqrt{2}}$

$S_3 = \sqrt{2 + S_2} = \sqrt{2 + \sqrt{2 + \sqrt{2}}}$, $S_4 = \sqrt{2 + S_3} = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}$

(ii) $S_1 = 1,\ S_2 = 1,\ S_{n+2} = S_{n+1} + S_n$, $n = 1, 2, 3 \cdots$ Terms in this sequence are

$S_3 = S_2 + S_1 = 1+1 = 2$, $S_4 = S_3 + S_2 = 2+1 = 3$, $S_5 = S_4 + S_3 = 3+2 = 5$, ----

This sequence $1, 1, 2, 3, 5, 8, 13, 21 \cdots$ is called the Fibonacci sequence

* Write the formula of S_n for the following sequences

(i) $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \cdots$

Sol: $S_n = \frac{1}{2n-1}$

$\frac{1}{1}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \cdots = \frac{1}{2\times1-1}, \frac{1}{2\times2-1}, \frac{1}{2\times3-1}, \frac{1}{2\times4-1} \cdots$

(ii) $1, 0, 1, 0, 1, 0 \cdots$

Sol: $S_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$

(or)

$S_n = \frac{1+(-1)^{n+1}}{2}$

(iii) $2, \frac{5}{2}, \frac{10}{3}, \frac{17}{4}, \frac{26}{5}, \cdots$

Sol: $S_1 = 2 = 1+1$

$S_2 = \frac{5}{2} = \frac{2\times2+1}{2} = 2 + \frac{1}{2}$

$S_3 = \frac{10}{3} = \frac{3\times3+1}{3} = 3 + \frac{1}{3}$

$S_4 = \frac{17}{4} = \frac{4\times4+1}{4} = 4 + \frac{1}{4}$

$S_5 = \frac{26}{5} = \frac{5\times5+1}{5} = 5 + \frac{1}{5}$

$S_n = n + \frac{1}{n}$

* Write the terms of the following sequences.

(i) $S_1 = \sqrt{6}$, $S_{n+1} = \sqrt{c S_n}$

Sol: The terms of this sequence are

$S_1 = \sqrt{c}$, $S_2 = \sqrt{c\sqrt{c}}$, $S_3 = \sqrt{c\sqrt{c\sqrt{c}}}$, ----

(ii) $S_n = \dfrac{12+5n}{11n+12}$

Sol: The terms of this sequence are

$$S_1 = \frac{12+5}{11+12} = \frac{17}{23}$$

$$S_2 = \frac{12+5(2)}{11(2)+12} = \frac{22}{34}$$

$$S_3 = \frac{12+5(3)}{11(3)+12} = \frac{12+15}{33+12} = \frac{27}{45} \text{ ----}$$

Subsequence: If $\{S_n\}$ is a sequence and $\{n_r\}$ is a sequence of positive integers $\ni n_1 < n_2 < n_3 < \cdots$ then $\{S_{n_r}\}$ is called a subsequence of $\{S_n\}$.

Ex: $\{\frac{1}{n^2}\}$ is a subsequence of sequence $\{\frac{1}{n}\}$

Here the sequence $\{S_n\} = \{\frac{1}{n}\}$ is $S_1 = 1$, $S_2 = \frac{1}{2}$, $S_3 = \frac{1}{3}$,

$S_4 = \frac{1}{4}$, ---

$n_1 = 2^2 = 4$, $n_2 = 3^2 = 9$, $n_3 =$

$n_1 = 1^2 = 1 < n_2 = 2^2 = 4, < n_3 = 3^2 = 9, < \cdots$

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$\therefore \{S_{n_r}\} = \left\{\frac{1}{n^2}\right\}$

Range of a sequence: The set of all terms of a sequence is called the range of the sequence.

Ex: Terms of the sequence $\{S_n\} = \{(-1)^n\}$ are

$S_1 = 1,\ S_2 = 1,\ S_3 = -1,\ S_4 = 1, \ldots$

Range of this sequence is $\{-1, 1\}$

Boundedness.

Boundedness of a sequence: A sequence S_n is said to be bounded if its range set is bounded.

(or)

A sequence S_n is bounded if ~~there exist~~

$\exists\ k_1, k_2 \in \mathbb{R} \ni k_1 \le S_n \le k_2 \ \forall\, n \in Z^+$

Ex: The sequence $\{S_n\} = \{(-1)^n\}$ is bounded because $-1 \le S_n \le 1 \ \forall\, n \in Z^+$

* Show that the sequence $\frac{3}{2}, -\frac{4}{3}, \frac{5}{4}, \ldots$ is bounded.

Sol: Here $S_n = \frac{(-1)^n \cdot n+1}{n}$, for $n = 2, 3, 4, \ldots$

$$|S_n| = \left|\frac{(-1)^n (n+1)}{n}\right| = \left|\frac{n+1}{n}\right| = \frac{n+1}{n} = 1 + \frac{1}{n} < 2$$

i.e., $|S_n| < 2 \Rightarrow -2 < S_n < 2$ for $n = 2, 3, 4, \ldots$

$S_1 = \frac{(-1)^1(1+1)}{1}$ (-2)

$\therefore -2 \leq S_n < 2 \;\forall\, n \in Z^+$

$\therefore$ The sequence S_n is bounded.

* Discuss the boundedness of the following sequences.

(i) $S_n = 1 + \frac{(-1)^n}{n}$

Sol: Here $S_1 = 1 + \frac{-1}{1} = 0$, $S_2 = 1 + \frac{1}{2} = \frac{3}{2}$, $S_3 = 1 + \frac{-1}{3} = \frac{2}{3}$,

$S_4 = 1 + \frac{1}{4} = \frac{5}{4}, \ldots$

Here $0 \leq S_n \leq \frac{3}{2} \;\forall\, n \in Z^+$

$\therefore$ The sequence S_n is bounded.

(ii) $S_n = (-1)^{n-1} n$

Sol: Here $S_1 = (-1)^{1-1} \cdot 1 = 1$

$S_2 = (-1)^{2-1} \cdot 2 = -2$

$S_3 = (-1)^{3-1} \cdot 3 = 3$

$S_4 = (-1)^{4-1} \cdot 4 = -4 \ldots$

For this sequence $\exists$ no $K_1, K_2 \in \mathbb{R} \ni K_1 \leq S_n \leq K_2$ the

$\therefore$ The sequence $\{(-1)^n n\}$ is not bounded.

* Limit of a sequence and convergence of a sequence:

Limit of a sequence: Let $\{S_n\}$ be a sequence and $l \in \mathbb{R}$. 'l' is said to be the limit of the sequence $\{S_n\}$ if to each $\varepsilon > 0$ $\exists$ $m \in \mathbb{Z}^+ \ni |S_n - l| < \varepsilon \ \forall n \geq m$

It is denoted by $\lim_{n \to \infty} S_n = l$

Convergence of a sequence: A sequence $\{S_n\}$ is said to be convergent. If ~~then~~ $\exists$ $l \in \mathbb{R} \ni$ for each $\varepsilon > 0$ $\exists$ $m \in \mathbb{Z}^+ \ni |S_n - l| < \varepsilon \ \forall n \geq m$ i.e., A sequence $\{S_n\}$ is said to be convergent iff it has the limit.

Divergent sequence: A sequence $\{S_n\}$ is said to be divergent if it not convergent.

* Show that the sequence $\{S_n\} = \{1/n\}$ converges to '0'.

Sol: Given that $S_n = 1/n$

let $\varepsilon > 0$

$$|S_n - 0| < \varepsilon \Rightarrow \left|\frac{1}{n} - 0\right| < \varepsilon$$

$$\Rightarrow \frac{1}{n} < \varepsilon$$

$$\Rightarrow n > \frac{1}{\varepsilon}$$

i.e., if $n > \frac{1}{\varepsilon}$ then $|S_n - 0| < \varepsilon$

If we choose $m = \left[\frac{1}{\varepsilon}\right] + 1 \in \mathbb{Z}^+$ then

$$|S_n - 0| < \varepsilon \ \forall n \geq m$$

$\therefore \lim_{n\to\infty} S_n = 0$

Hence $\{\frac{1}{n}\}$ converges to '0'.

* Show that the sequence $\{S_n\}$ defined by $S_n = \frac{3n-1}{4n+5}$ c
to $\frac{3}{4}$.

Sol: Given that $S_n = \frac{3n-1}{4n+5}$

let $\varepsilon > 0$

$$\left|S_n - \frac{3}{4}\right| < \varepsilon \Rightarrow \left|\frac{3n-1}{4n+5} - \frac{3}{4}\right| < \varepsilon$$

$$\left|\frac{\cancel{12n}-4-\cancel{12n}-15}{4(4n+5)}\right| < \varepsilon$$

$$\Rightarrow \left|\frac{-19}{16n+20}\right| < \varepsilon$$

$$\Rightarrow \frac{19}{4(4n+5)} < \varepsilon$$

$$\Rightarrow \frac{1}{4n+5} < \frac{4\varepsilon}{19}$$

$$\Rightarrow 4n+5 > \frac{19}{4\varepsilon}$$

$$\Rightarrow 4n > \frac{19}{4\varepsilon} - 5$$

$$\Rightarrow n > \frac{19}{16\varepsilon} - \frac{5}{4}$$

i.e., If $n > \frac{19}{16\varepsilon} - \frac{5}{4}$ then $\left|S_n - \frac{3}{4}\right| < \varepsilon$

If we choose $m = \left[\frac{19}{16\varepsilon} - \frac{5}{4}\right] + 1 \in Z^+$ then

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$\left|S_n - \frac{3}{4}\right| < \varepsilon \ \forall n \geq m.$

$\therefore \lim_{n\to\infty} S_n = \frac{3}{4}$

Hence the $\left\{\frac{3n-1}{4n+5}\right\}$ converges to $\frac{3}{4}$.

* If $S_n = \sqrt{n+1} - \sqrt{n}$ then prove that $\lim S_n = 0$

Sol: Given that $S_n = \sqrt{n+1} - \sqrt{n}$

$$= \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}}$$

$\sqrt{n+1} > \sqrt{n}$

$\sqrt{n+1} + \sqrt{n} > \sqrt{n} + \sqrt{n}$

$\frac{1}{\sqrt{n+1} + \sqrt{n}} > \frac{1}{\sqrt{n} + \sqrt{n}}$

$$= \frac{\cancel{n}+1-\cancel{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1}\ \sqrt{n}} < \frac{1}{\sqrt{n} + \sqrt{n}} = \frac{1}{2\sqrt{n}}$$

let $\varepsilon > 0$ &

$|S_n - 0| < \frac{1}{2\sqrt{n}} - 0 < \varepsilon$

$\Rightarrow \frac{1}{2\sqrt{n}} < \varepsilon \Rightarrow 2\sqrt{n} > \frac{1}{\varepsilon}$

$\Rightarrow \sqrt{n} > \frac{1}{2\varepsilon} \Rightarrow n > \frac{1}{4\varepsilon^2}$

i.e., if $n > \frac{1}{4\varepsilon^2}$ then $|S_n - 0| < \varepsilon$

If we choose $m = \left[\frac{1}{4\varepsilon^2}\right] + 1$ then $|S_n - 0| < \varepsilon \ \forall n \geq m.$

$\therefore \lim_{n\to\infty} S_n = 0$

A-1

Imp * Prove that $\{\sqrt[n]{n}\}$ converges to 1.

Sol: ~~Given that~~ let $S_n = \sqrt[n]{n} - 1 = n^{1/n} - 1$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} [n^{1/n} - 1] = 0$

let $\varepsilon > 0$

$S_n = n^{1/n} - 1 \Rightarrow n^{1/n} = 1 + S_n$

$$\Rightarrow n = (1+s_n)^n = 1 + ns_n + \frac{n(n-1)}{2!} s_n^2 + \cdots + s_n^n \geq$$

$$\Rightarrow n \geq \frac{n(n-1)}{2!} s_n^2$$

$$\Rightarrow \frac{2n}{n(n-1)} \geq s_n^2 \Rightarrow s_n^2 \leq \frac{2}{n-1} \Rightarrow s_n \leq \sqrt{\frac{2}{n-1}}$$

let $s_n \leq \sqrt{\frac{2}{n-1}} < \varepsilon$

$$\Rightarrow \sqrt{\frac{2}{n-1}} < \varepsilon$$

$$\Rightarrow \frac{2}{n-1} < \varepsilon^2$$

$$\Rightarrow \frac{1}{n-1} < \frac{\varepsilon^2}{2} \Rightarrow n-1 > \frac{2}{\varepsilon^2}$$

$$\Rightarrow n > \frac{2}{\varepsilon^2} + 1$$

i.e., if $n > \frac{2}{\varepsilon^2} + 1$ then $s_n \leq \sqrt{\frac{2}{n-1}} < \varepsilon$

$\therefore$ If we choose $m = \left[\frac{2}{\varepsilon^2} + 1\right] + 1 \in \mathbb{Z}^+$ then

$$|s_n - 0| < \varepsilon \ \forall \ n \geq m$$

$$\Rightarrow \lim_{n\to\infty} s_n = 0 \Rightarrow \lim_{n\to\infty} \left[\sqrt[n]{n} - 1\right] = 0$$

$$\Rightarrow \lim_{n\to\infty} \sqrt[n]{n} = 1$$

Hence the $\{\sqrt[n]{n}\}$ converges to 1.

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* Theorem (uniqueness of limit):

A convergent sequence has unique limit.

Proof: Let S_n be a convergent sequence. Let l, l' be two limits of the sequence $\{S_n\}$

let $\varepsilon = \frac{|l-l'|}{2}$

$\{S_n\}$ converges to $l \Rightarrow$ for $\varepsilon > 0 \ \exists \ m_1 \in Z^+ \ni |S_n - l| < \varepsilon \ \forall n \geq m_1$

$\{S_n\}$ converges to $l' \Rightarrow$ for $\varepsilon > 0 \ \exists \ m_2 \in Z^+ \ni |S_n - l'| < \varepsilon \ \forall n \geq m_2$

let $m = \max\{m_1, m_2\}$

Then $|S_n - l| < \varepsilon \ \forall n \geq m$ &

$|S_n - l'| < \varepsilon \ \forall n \geq m$

consider $|l - l'| = |l - S_n + S_n - l'|$

$\leq |S_n - l| + |S_n - l'|$

$< \varepsilon + \varepsilon \ \forall n \geq m$

$< 2\varepsilon$

$< 2 \frac{|l-l'|}{2}$

i.e., $|l - l'| < |l - l'|$

This is a contridiction

$\therefore$ A convergent sequence has unique limit.

Note: If $\{S_n\}$ is a sequence of positive terms and $\lim_{n\to\infty} S_n = l$ then $l \geq 0$

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* Theorem: If the $\{S_n\}$ converges to 'l' then any subsequence of S_n is also converges to 'l'.

Proof:- Let the $\{S_n\}$ be converges to 'l'.

Then to each $\varepsilon > 0 \; \exists \; m \in Z^+ \ni |S_n - l| < \varepsilon \; \forall n \geq m$.

Let $\{u_n\}$ be a subsequence of $\{S_n\}$, then $\exists$ a sequence positive integers $\{r_n\} \ni r_1 < r_2 < \cdots$ & $u_n = S_{r_n}$.

$\because \{r_n\}$ is a sequence of positive integers, $r_1 \geq 1$

$\because r_1 < r_2 < \cdots$, by induction $r_n \geq n$

$\therefore$ For $n \geq m$, $r_n \geq n \geq m$ where we have

$$|S_n - l| < \varepsilon \Rightarrow \text{For } r_n \geq m, |S_{r_n} - l| < \varepsilon$$

$$\Rightarrow |u_n - l| < \varepsilon \; \forall n \geq m$$

$\therefore$ The sequence $\{u_n\}$ converges to 'l'. Hence every subsequence of $\{S_n\}$ converges to 'l'.

* Theorem: If the subsequences $\{S_{2n-1}\}$, $\{S_{2n}\}$ of a sequence $\{S_n\}$ converges to the same limit 'l' then the sequence $\{S_n\}$ is also converges to l.

Proof: Let $\{S_{2n-1}\}$, $\{S_{2n}\}$ be two subsequences of $\{S_n\}$ and converges to 'l'.

Let $\varepsilon > 0$

$\{S_{2n-1}\}$ converges to $l \Rightarrow$ For $\varepsilon < 0 \; \exists \; m_1 \in Z^+ \ni |S_{\dots} - l| <$

$\{S_{2n}\}$ converges to $l \Rightarrow$ For $\varepsilon > 0$ $\exists m_2 \in Z^+ \ni |S_{2n} - l| < \varepsilon$ $\forall$ $n \geq m_2$

let $m = \max\{m_1, m_2\}$. Then $|S_{2n-1} - l| < \varepsilon$, $|S_{2n} - l| < \varepsilon$ $\forall$ $n \geq m$

$$\Rightarrow |S_n - l| < \varepsilon \ \forall n \geq m$$

$\therefore$ The sequence $\{S_n\}$ converges to 'l'.

* Prove that the sequence $\left\{\frac{(-1)^n}{n}\right\}$ converges to 0.

Sol: Given that the sequence $S_n = \left\{\frac{(-1)^n}{n}\right\}$. The terms in this sequence are $S_1 = -1$, $S_2 = \frac{1}{2}$, $S_3 = \frac{-1}{3}$, $S_4 = \frac{1}{4}$, $S_5 = -\frac{1}{5}$, ...

Here the subsequence $\{S_{2n-1}\}$ is $\left\{\frac{-1}{2n-1}\right\} = -1, -\frac{1}{3}, \frac{-1}{5}$ ...

The subsequence $\{S_{2n}\}$ is $\left\{\frac{1}{2n}\right\} = \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots$

Here the subsequences $\{S_{2n-1}\}$ & $\{S_{2n}\}$ converges to '0'.

Hence the sequence $\{S_n\} = \left\{\frac{(-1)^n}{n}\right\}$ converges to '0'.

* Prove that the sequence $\{S_n\} = \{(-1)^n\}$ does not converge

Sol: Given that $\{S_n\} = \{(-1)^n\}$

i.e., $S_1 = -1$, $S_2 = 1$, $S_3 = -1$, $S_4 = 1$ ....

Here the subsequence $\{S_{2n-1}\}$ is $\{(-1)^{2n-1}\} = -1, -1, -1 \ldots$

$= \{-1\}$ converges to '-1'

The subsequence $\{S_{2n}\} = \{1\}$ converges to '1'.

Hence the sequence $\{S_n\}$ does not converge.

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* Algebra of convergent sequences: (1) If $\{S_n\}$ & $\{t_n\}$ are two convergent sequences and limit $\lim S_n = l$, $\lim t_n = l'$ then $\lim(S_n + t_n) = l + l'$

(2) If $\lim S_n = l$ & $c \in \mathbb{R}$ then $\lim(c S_n) = cl$.

(3) If $\lim S_n = l$, $\lim t_n = l'$ then $\lim(S_n t_n) = ll'$ & $\lim(S_n - t_n) = l - l'$

(4) If $\lim S_n = l$, $\lim t_n = l'$, $t_n \neq 0 \ \forall n \in \mathbb{Z}^+$ & $l' \neq 0$ then

$\lim \frac{S_n}{t_n} = \frac{l}{l'}$

As - 1

Imp * Theorem: Every convergent sequence is bounded.

Proof: Let $\{S_n\}$ be a convergent sequence & $\lim S_n = l$.

Let $\varepsilon = 1$, then for $\varepsilon = 1 > 0 \ \exists \ m \in \mathbb{Z}^+ \ni |S_n - l| < 1 \ \forall n \geq m$

$\Rightarrow -1 < S_n - l < 1 \ \forall n \geq m$

$\Rightarrow l - 1 < S_n < l + 1$ — (1) $\forall n \geq m$

let $K_1 = \min\{l-1, S_1, S_2, \cdots S_{m-1}\}$ &

$K_2 = \max\{l+1, S_1, S_2, \cdots S_{m-1}\}$

Then $K_1 \leq l - 1 < S_n < l + 1 \leq K_2 \ \forall n \geq m$ &

$K_1 \leq S_1, S_2, \cdots S_{m-1} \leq K_2$

$\Rightarrow K_1 \leq S_n \leq K_2 \ \forall n \in \mathbb{Z}^+$

$\therefore$ The sequence $\{S_n\}$ is bounded.

1M gmp

Note: Converse of this theorem is not true i.e, every bounded sequence need not be convergent.

For example, consider a sequence $\{S_n\} = \{(-1)^n\}$. Terms of this sequence are $S_1 = -1, S_2 = +1, \ldots$

$\because -1 \leq S_n \leq +1 \ \forall n \in Z^+$, $\{S_n\}$ is bounded.

Here a subsequence $\{S_{2n-1}\}$, $-1, -1, \ldots$ converges to '-1' & a subsequence $\{S_{2n}\}$, $1, 1, \ldots$ converges to '$+1$'.

Hence the sequence $\{S_n\}$ is not convergent.

* **Theorem:** If a sequence $\{S_n\}$ converges to 'l' then the sequence $\{|S_n|\}$ converges to $|l|$.

(or)

If $\lim S_n = l$ then $\lim |S_n| = |l|$

Proof: Let the sequence $\{S_n\}$ be converges to 'l' and $\varepsilon > 0$, then for $\varepsilon > 0 \ \exists \ m \in Z^+ \ni |S_n - l| < \varepsilon \ \forall n \geq m$.

We know that $||a| - |b|| \leq |a - b|$ for any $a, b \in \mathbb{R}$

$$\therefore \left| |S_n| - |l| \right| \leq |S_n - l| < \varepsilon \ \forall n \geq m$$

$$\Rightarrow \text{for } \varepsilon > 0 \ \exists \ m \in Z^+ \ni ||S_n| - |l|| < \varepsilon \ \forall n \geq m$$

$$\therefore \lim |S_n| = |l|$$

Hence the sequence $\{|S_n|\}$ converges to $|l|$

Note: Converse of this theorem is not true i.e., $\{|S_n|\}$ converges to $|l|$ then $\{S_n\}$ need not be converge

For example, consider $\{s_n\} = \{(-1)^n\}$ is not converg[ent]
but $\{|s_n|\} = \{|(-1)^n|\} = \{1\}$, 1,1,1,--- converges to 1.

v.Imp
* Sandwich theorem (or) squeeze theorem: If $\{s_n\}$, $\{t_n\}$,
AS-1 $\{u_n\}$ are three sequences such that

(i) $s_n \le u_n \le t_n$ for $n \ge k$, where k is some posi[tive] integer and

(ii) $\lim s_n = \lim t_n = l$, then $\lim u_n = l$.

proof: Let $\{s_n\}, \{t_n\}, \{u_n\}$ be three sequences such that

(i) $s_n \le u_n \le t_n$ —(1) for $n \ge k$, where k is some positive integ[er]

(ii) $\lim s_n = \lim t_n = l$ ~~then $\lim u_n = l$~~

Now we have to prove $\lim u_n = l$

let $\varepsilon > 0$.

$\lim s_n = l \Rightarrow$ for $\varepsilon > 0\ \exists\ m_1 \in Z^+ \ni |s_n - l| < \varepsilon\ \forall\ n \ge m_1$

$\Rightarrow -\varepsilon < s_n - l < \varepsilon\ \forall\ n \ge m_1$

$\Rightarrow l - \varepsilon < s_n < l + \varepsilon\ \forall\ n \ge m_1$ —(2)

$\lim t_n = l \Rightarrow$ for $\varepsilon > 0\ \exists\ m_2 \in Z^+ \ni |t_n - l| < \varepsilon\ \forall\ n \ge m_2$

$\Rightarrow -\varepsilon < t_n - l < \varepsilon\ \forall\ n \ge m_2$

$\Rightarrow l - \varepsilon < t_n < l + \varepsilon$ —(3) $\forall\ n \ge m_2$

Let $m = \max\{k, m_1, m_2\}$

Then the inequalities (1), (2), (3) holds for all $n \ge m$

* Now, from (1), (2) & (3)

$l-\varepsilon < S_n \le u_n \le t_n < l+\varepsilon \ \forall\ n \ge m$

$\Rightarrow l-\varepsilon < u_n < l+\varepsilon \ \forall\ n \ge m$

$\Rightarrow |u_n - l| < \varepsilon \ \forall\ n \ge m$

ie., for $\varepsilon > 0$ $\exists$ $m \in Z^+ \ni |u_n - l| < \varepsilon \Rightarrow \lim u_n = l$.

* **Theorem:** If $\lim\limits_{n\to\infty}\left[\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}\right] = 1$

AS - 1

Proof: Let $S_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}$

For $1 \le m \le n$, $n^2+1 \le n^2+m \le n^2+n$

$$\Rightarrow \sqrt{n^2+1} \le \sqrt{n^2+m} \le \sqrt{n^2+n}$$

$$\Rightarrow \frac{1}{\sqrt{n^2+1}} \ge \frac{1}{\sqrt{n^2+m}} \ge \frac{1}{\sqrt{n^2+n}} \quad \text{--- (1)}$$

Putting $m = 1, 2, 3, \ldots n$ in (1) we get

$m=1$, $\frac{1}{\sqrt{n^2+1}} \ge \frac{1}{\sqrt{n^2+1}} \ge \frac{1}{\sqrt{n^2+n}}$

$m=2$, $\frac{1}{\sqrt{n^2+1}} \ge \frac{1}{\sqrt{n^2+2}} \ge \frac{1}{\sqrt{n^2+n}}$

$m=3$, $\frac{1}{\sqrt{n^2+1}} \ge \frac{1}{\sqrt{n^2+3}} \ge \frac{1}{\sqrt{n^2+n}}$

$\vdots$

$m=n$, $\frac{1}{\sqrt{n^2+1}} \ge \frac{1}{\sqrt{n^2+n}} \ge \frac{1}{\sqrt{n^2+n}}$

Adding these inequalities we get

$$\frac{n}{\sqrt{n^2+1}} \ge \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots \frac{1}{\sqrt{n^2+n}} \ge \frac{n}{\sqrt{n^2+n}}$$

$$\frac{n}{n\sqrt{1+\frac{1}{n^2}}} \geq S_n \geq \frac{n}{n\sqrt{1+1/n}}$$

$$\Rightarrow \frac{1}{\sqrt{1+\frac{1}{n}}} \leq S_n \leq \frac{1}{\sqrt{1+\frac{1}{n^2}}}$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}} = 1 \ \& \ \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}} = 1$$

$\therefore$ By sandwich theorem $\lim_{n\to\infty} S_n = 1$

i.e., $\lim_{n\to\infty}\left[\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots \frac{1}{\sqrt{n^2+n}}\right] = 1$

* Using sandwich theorem prove that

$$\lim_{n\to\infty}\left[\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots \frac{1}{(n+n)^2}\right] = 0$$

Sol: let $S_n = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots \frac{1}{(n+n)^2}$

For $1 \leq m \leq n \Rightarrow n+1 \leq n+m \leq n+n$

$$\Rightarrow (n+1)^2 \leq (n+m)^2 \leq (n+n)^2$$

$$\Rightarrow \frac{1}{(n+1)^2} \geq \frac{1}{(n+m)^2} \geq \frac{1}{(n+n)^2} \quad \text{—(1)}$$

putting $m = 1, 2, 3, \cdots n$ we get

$m=1$, $\frac{1}{(n+1)^2} \geq \frac{1}{(n+1)^2} \geq \frac{1}{(n+n)^2}$

$m=2$, $\frac{1}{(n+1)^2} \geq \frac{1}{(n+2)^2} \geq \frac{1}{(n+n)^2}$

$m=3$, $\frac{1}{(n+1)^2} \geq \frac{1}{(n+3)^2} \geq \frac{1}{(n+n)^2}$

$\vdots$

$m=n$, $\frac{1}{(n+1)^2} \geq \frac{1}{(n+n)^2} \geq \frac{1}{(n+n)^2}$

Adding these inequalities we get $\qquad n+1 \geq n$, $\frac{1}{(n+1)^2} \leq \frac{1}{n^2}$

$\cancel{n(n+1)^2 \leq S_n \leq}$

$$\frac{n}{(n+1)^2} \geq S_n \geq \frac{n}{(n+n)^2} \Rightarrow \cancel{\frac{n}{4n^2} \leq S_n \leq \frac{n}{(n+1)^2} \leq \frac{n}{n^2}}$$

$$\cancel{\lim_{n\to\infty} \frac{n}{(n+1)^2} = 0} \ \& \ \cancel{\lim_{n\to\infty} \frac{n}{(n+n)^2} = 0}$$

$\therefore$ ~~By sandwich theorem~~ $\cancel{\lim_{n\to\infty} S_n = 0}$

$$\Rightarrow \frac{n}{(2n)^2} \leq S_n \leq \frac{n}{(n+1)^2}$$

$$\Rightarrow \frac{\cancel{n}}{4n^{\cancel{2}}} \leq S_n \leq \frac{n}{(n+1)^2} \leq \frac{\cancel{n}}{n^{\cancel{2}}}$$

$$\lim_{n\to\infty} \frac{1}{4n} = 0 \ \& \ \lim_{n\to\infty} \frac{1}{n} = 0$$

$\therefore$ By sandwich theorem $\lim_{n\to\infty} S_n = 0$

Hence $\lim_{n\to\infty} \left[\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots \frac{1}{(n+n)^2}\right] = 0.$

Imp

*** Discuss the nature of the sequence $\{r^n\}$ for all $-1 < r_n \leq 1$.**

Sol:- Given sequence is $\{r^n\}$, we have to test the convergence of the sequence $\{r^n\}$ $\forall -1 < r \leq 1$

Case(i): Let $r=1$, then $r^n = 1 \ \forall n \in Z^+$

$\therefore$ The sequence $\{r^n\} = \{1^n\}$ converges to 1.

Case(ii): Let $0 < r < 1$

Put $r = \frac{1}{[illegible]}$ where $h > 0$