Differentiation and Mean Value Theorem

Let $f: S \to \mathbb{R}$ be a function. Let $a \in S$ be a limit point of S, and $l \in \mathbb{R}$. f is said to be differentiable at a if for each $\varepsilon > 0 \ \exists \ \delta > 0 \ \ni x \in S$, $|x-a| < \delta \Rightarrow \left| \frac{f(x)-f(a)}{x-a} - l \right| < \varepsilon$

Here 'l' is said to be derivative of f at 'a' and is denoted by $f'(a)$

i.e., If $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ exists then we say that f is derivable at a & is denoted by $f'(a)$

If $\lim_{x \to a^-} \frac{f(x)-f(a)}{x-a}$ exists then we say that f is differentiable from left. and that limit is denoted by $Lf'(a)$, known as left hand derivative of f at 'a'.

If $\lim_{x \to a^+} \frac{f(x)-f(a)}{x-a}$ exists then we say that f is derivable from right at 'a' and that limit is denoted by $Rf'(a)$, known as right hand derivative of f at 'a'.

Note: $Lf'(a) = Rf'(a) \Longleftrightarrow f'(a)$ exists.

Derivability on $[a,b]$: A function $f: [a,b] \to \mathbb{R}$ is said to be derivable on $[a,b]$ if

(i) $Rf'(a)$ exists

(ii) Derivable on (a,b) i.e., $f'(c)$ exists for every $c\in(a,b)$

(iii) $Lf'(a)$ exists.

*show that $f(x)=\sin x$ is derivable at every $a\in\mathbb{R}$.

Sol: Given that $f(x)=\sin x \ \forall\ x\in\mathbb{R}$

let $a\in\mathbb{R}$

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{\sin x-\sin a}{x-a}$$

$$=\lim_{x\to a}\frac{2\sin\left(\frac{x-a}{2}\right)\cdot\cos\left(\frac{x+a}{2}\right)}{x-a}$$

$$=\lim_{x\to a}\frac{2\sin\left(\frac{x-a}{2}\right)}{x-a}\cdot\lim_{x\to a}\cos\left(\frac{x+a}{2}\right)$$

$$=\lim_{\frac{x-a}{2}\to 0}\frac{\sin\left(\frac{x-a}{2}\right)}{\left(\frac{x-a}{2}\right)}\cdot\cancel{\lim}\cos\left(\frac{a+a}{2}\right)$$

$$\left[\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1\right]$$

$$=1\cdot\cos a=\cos a$$

$\therefore f(x)=\sin x$ is derivable at 'a' and $f'(a)=\cos a$

$\because$ 'a' is an arbitary element in $\mathbb{R}$, f is derivable on $\mathbb{R}$

Scanned with CamScanner

Imp (5m) ✓

Theorem: If $f:[a,b]\to\mathbb{R}$ is deriable at $c\in[a,b]$, then f is continuous at c.

proof: Let $f:[a,b]\to\mathbb{R}$ be ~~continuous~~ derivable at $c\in[a,b]$

let $c\in(a,b)$ then f is derivable at c.

$$\Rightarrow \lim_{x\to c}\frac{f(x)-f(c)}{x-c} = f'(c) \text{ exists}$$

For $x\neq c$, $f(x)-f(c) = \dfrac{f(x)-f(c)}{x-c}(x-c)$

$$\Rightarrow \lim_{x\to c} f(x)-f(c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}\cdot\lim_{x\to c}(x-c)$$

$$= f'(c)\cdot 0$$

$$= 0$$

$\therefore \lim_{x\to c} f(x) = f(c)$

$\therefore f$ is continuous at $c\in(a,b)$

Let $c=a$, then f is ~~right~~ derivable from right at $c=a$

$$\Rightarrow \lim_{x\to a^+}\frac{f(x)-f(a)}{x-a} = Rf'(a) \text{ exists.}$$

For $x\neq a$

$$f(x)-f(a) = \frac{f(x)-f(a)}{x-a}(x-a)$$

$$\Rightarrow \lim_{x\to a^+} f(x)-f(a) = \lim_{x\to a^+}\frac{f(x)-f(a)}{x-a}\cdot\lim_{x\to a^+}(x-a)$$

$$= Rf'(a)\cdot 0 = 0$$

$$\lim_{x \to a^+} f(x) = f(a)$$

f is continuous from right at a

Similarly,

we can prove that f is continuous from left at b

Hence f is continuous at $c \in [a,b]$

Note:

Converse of this theorem is not true i.e., every continuous function need not be differentiable

Consider a function $f(x) = |x| \ \forall x \in \mathbb{R}$

$f(x) = |x|$ is defined as $f(x) = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0} (-x) = 0$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0} (x) = 0$$

$$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

f is continuous at $x = 0$.

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{-x - 0}{x} = \lim_{x \to 0} -\frac{x}{x} = -1$$

$$= Lf'(0)$$

$$\Rightarrow \lim_{x \to a^+} f(x) = f(a)$$

f is continuous from right at a

Similarly,

we can prove that f is continuous from left at b

Hence f is continuous at $c \in [a,b]$

Note:

Converse of this theorem is not true i.e., every continuous function need not be differentiable

Consider a function $f(x) = |x| \; \forall x \in \mathbb{R}$

$f(x) = |x|$ is defined as $f(x) = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0} (-x) = 0$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0} (x) = 0$$

$$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

f is continuous at $x=0$.

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{-x-0}{x} = \lim_{x \to 0} -\frac{x}{x} = -1$$

$$= Lf'(0)$$

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x-0}{x} = \frac{x}{x} = 1 = Rf'(0)$$

$\because Lf'(0) \neq Rf'(0)$, f is not derivable at $x=0$

* Show that $f(x) = x\sin(\frac{1}{x})$ for $x \neq 0$, $f(x)=0$ for $x=0$ is continuous but not derivable at $x=0$

Sol: Given function is $f(x) = \begin{cases} x\sin(\frac{1}{x}) & \text{for } x\neq 0 \\ 0 & \text{for } x=0 \end{cases}$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left[x\cdot\sin\frac{1}{x}\right] = 0 \quad \because \lim_{x\to 0} x = 0 \text{ \& } \sin\frac{1}{x} \text{ is bounded}$$

ie, $\lim_{x\to 0} f(x) = 0 = f(0)$

$\therefore f$ is continuous at $x=0$

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x\cdot\sin\frac{1}{x} - 0}{x-0}$$

$$= \lim_{x\to 0} \frac{\cancel{x}\sin\frac{1}{x}}{\cancel{x}}$$

$$= \lim_{x\to 0} \sin\frac{1}{x} \text{ does not exist}$$

$\therefore f$ is not derivable at $x=0$

* Examine the continuity and differentiability at $f(x) = |x-1| + |x-2|$ at $x=1,2$.

Sol: Given $f(x) = |x-1| + |x-2|$

If $x<1$ then $x-1<0 \Rightarrow |x-1| = -(x-1) = 1-x$

If $x<1<2$ then $x-2<0 \Rightarrow |x-2| = -(x-2) = 2-x$

$\therefore$ If $x<1$, $f(x) = |x-1| + |x-2| = 1-x+2-x = 3-2x$

If $1<x<2$ then $x-1>0$ & $x-2<0$

$\Rightarrow |x-1| = x-1$ & $|x-2| = -(x-2) = 2-x$

$\therefore$ If $1<x<2$, $f(x) = |x-1| + |x-2| = x-1+2-x$
$= 1$

If $x>2>1$ then $x-1>0$, $x-2>0$

$\Rightarrow |x-1| = x-1$, $|x-2| = x-2$

$\therefore$ If $x>2$ then $f(x) = |x-1| + |x-2|$
$= x-1+x-2 = 2x-3$

$f(1) = |1-1| + |1-2| = 0 + |-1| = 1$

$f(2) = |2-1| + |2-2| = 1+0 = 1$

$\therefore$ The given function is

$$f(x) = \begin{cases} 3-2x & \text{if } x<1 \\ 1 & \text{if } 1 \le x \le 2 \\ 2x-3 & \text{if } x>2 \end{cases}$$

(i) continuity at $x=1$:

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1} (3-2x) = 1$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1} (1) = 1$

$\therefore \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$

Scanned with CamScanner

f is continuous at '1'

Differentiability at $x=1$

$$\lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{3-2x-1}{x-1} = \lim_{x\to 1} \frac{2(1-x)}{x-1}$$

$$= -2 = Lf'(1)$$

$$\lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{1-1}{x-1} = 0 = Rf'(1)$$

$\because Lf'(1) \neq Rf'(1)$, f is not derivable at $x=1$

(ii) continuity at $x=2$:

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2} 1 = 1$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2} (2x-3) = 1$$

$$\therefore \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$$

f is continuous at '2'

Differentiability at $x=2$.

$$\lim_{x\to 2^-} \frac{f(x)-f(2)}{x-2} = \frac{1-1}{x-2} = 0 = Lf'(2)$$

$$\lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2} = \frac{2x-3-1}{x-2} = \frac{2x-4}{x-2} = \frac{2\cancel{(x-2)}}{\cancel{x-2}}$$

$$= 2 = Rf'(2)$$

$\because Lf'(2) \neq Rf'(2)$, f is not differentiable at $x=2$

Scanned with CamScanner

* If $f(x) = \frac{x}{1+e^{1/x}}$ if $x \neq 0$, and $f(0)=0$, show that f is continuous at '0' but not derivable at 0.

Sol: Given function $f(x) = \frac{x}{1+e^{1/x}}$ if $x \neq 0$ & $f(0)=0$

Continuity at $x=0$:

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} (0-h) = \lim_{h\to 0} f(-h) = \lim_{h\to 0} \frac{-h}{1+e^{-1/h}} = \frac{0}{1+0} = 0$$

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} (0+h) = \lim_{h\to 0} f(h) = \lim_{h\to 0} \frac{h}{1+e^{1/h}}$$

$$= \lim_{h\to 0} \frac{h \cdot e^{-1/h}}{e^{-1/h}+1} = \frac{0}{0+1} = 0$$

$\because \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$, f is continuous at $x=0$

Differentiability at $x=0$:

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{\frac{x}{1+e^{1/x}} - 0}{x-0} = \lim_{x\to 0^-} \frac{\cancel{x}}{(1+e^{1/x})\cdot \cancel{x}}$$

$$= \lim_{h\to 0} \frac{1}{1+e^{1/(0-h)}}$$

$$= \lim_{h\to 0} \frac{1}{1+e^{-1/h}} = 1 = Lf'(0)$$

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{\frac{x}{1+e^{1/x}} - 0}{x-0} = \lim_{x\to 0^+} \frac{\cancel{x}}{(1+e^{1/x})\cancel{x}}$$

$$= \lim_{h\to 0} \frac{1}{1+e^{1/(0+h)}}$$

$$= \lim_{h\to 0} \frac{1}{1+e^{1/h}} = \lim_{h\to 0} \frac{e^{-1/h}}{e^{-1/h}+1}$$

$$= \frac{0}{0+1} = 0 = Rf'(0)$$

$\therefore$ & $Lf'(0) \neq Rf'(0)$

f is not derivable at $x=0$.

Hence f is continuous but not derivable at $x=0$.

If $f(x) = x\left(\frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}\right)$ for $x \neq 0$ & $f(0)=0$ is not derivable at $x=0$

Given function is $f(x) = x\left(\frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}\right)$ for $x \neq 0$ & $f(0)=0$

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} \Rightarrow \lim_{x\to 0^-} \frac{x\left(\frac{e^{1/x} - e^{-1/x}}{e^{1/x}+e^{-1/x}}\right) - 0}{x-0}$$

$$= \lim_{x\to 0^-} \frac{\cancel{x}\left(\frac{e^{1/x} - e^{-1/x}}{e^{1/x}+e^{-1/x}}\right)}{\cancel{x}}$$

$$= \lim_{x\to 0^-} \frac{e^{1/x} - e^{-1/x}}{e^{1/x}+e^{-1/x}}$$

$$\cancel{= \lim_{h\to 0} e^{1/x}(0-1)}$$

$$= \lim_{h\to 0} \frac{e^{1/(0-h)} - e^{-(1/(0-h))}}{e^{1/(0-h)} + e^{-1/(0-h)}}$$

$$= \lim_{h\to 0} \frac{e^{-1/h} - e^{1/h}}{e^{-1/h} + e^{1/h}} = \lim_{h\to 0} \frac{e^{-1/h}(e^{1/h} - e^{1/h})}{e^{-1/h}(e^{-1/h}+e^{1/h})}$$

$$= \lim_{h\to 0} \frac{e^{-2/h} - 1}{e^{-2/h}+1} = \frac{-1}{1} = -1$$

$$= Lf'(0)$$

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{x\left(\frac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}\right)-0}{x-0}$$

$$= \lim_{x\to 0^+} \frac{\cancel{x}\left(\frac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}\right)}{\cancel{x}}$$

$$= \lim_{x\to 0^+} \frac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}$$

$$= \lim_{h\to 0} \frac{e^{1/(0+h)}-e^{-1/(0+h)}}{e^{1/(0+h)}+e^{-1/(0+h)}}$$

$$= \lim_{h\to 0} \frac{e^{1/h}-e^{-1/h}}{e^{1/h}+e^{-1/h}}$$

$$= \lim_{h\to 0} \frac{e^{-1/h}\left(e^{1/h}-e^{-1/h}\right)}{e^{-1/h}\left(e^{1/h}+e^{-1/h}\right)}$$

$$= \lim_{h\to 0} \frac{1-e^{-2/h}}{1+e^{-2/h}} = \frac{1}{1} = 1 = Rf'(0)$$

$\therefore Lf'(0) \neq Rf'(0)$

$\therefore f$ is not derivable at $x=0$.

* Show that $f(x) = x\left(\frac{e^{1/x}-1}{e^{1/x}+1}\right)$ if $x \neq 0$ & $f(0)=0$ is continuous but not derivable at $x=0$.

Sol: Given function is $f(x) = x\left(\frac{e^{1/x}-1}{e^{1/x}+1}\right)$ if $x\neq 0$ & $f(0)=0$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h)$$

$$\lim_{h\to 0} f(-h) = \lim_{h\to 0} -h\left(\frac{e^{-1/h}-1}{e^{-1/h}+1}\right) = \frac{0(0-1)}{0+1} = 0$$

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) = \lim_{h\to 0} \frac{h(e^{1/h}-1)}{e^{1/h}+1}$$

$$= \lim_{h\to 0} \frac{h\cdot e^{1/h}(1-e^{-1/h})}{e^{1/h}(1+e^{-1/h})} = \frac{0(1-0)}{0(1+0)} = 0$$

$$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 0$$

$\therefore$ f is continuous at $x=0$.

Differentiability at $x=0$.

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{x\left(\frac{e^{1/x}-1}{e^{1/x}+1}\right)-0}{x-0}$$

$$= \lim_{x\to 0^-} \frac{\cancel{x}\left(\frac{e^{1/x}-1}{e^{1/x}+1}\right)}{\cancel{x}}$$

$$= \lim_{x\to 0^-} \left(\frac{e^{1/x}-1}{e^{1/x}+1}\right)$$

$$= \lim_{h\to 0} \left(\frac{e^{1/(0-h)}-1}{e^{1/(0-h)}+1}\right)$$

$$= \lim_{h\to 0} \frac{e^{-1/h}-1}{e^{-1/h}+1} = \frac{0-1}{0+1} = \frac{-1}{1} = -1 = Lf'(0)$$

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{x\left(\frac{e^{1/x}-1}{e^{1/x}+1}\right)-0}{x-0} = \lim_{x\to 0^+} \frac{\cancel{x}\left(\frac{e^{1/x}-1}{e^{1/x}+1}\right)}{\cancel{x}}$$

Scanned with CamScanner

$$= \lim_{x\to 0^+} \frac{e^{1/x}-1}{e^{1/x}+1}$$

$$= \lim_{x\to 0^+} \frac{e^{1/(0+h)}-1}{e^{1/(0+h)}+1} = \lim_{x\to 0^+} \frac{e^{1/h}-1}{e^{1/h}+1}$$

$$= \lim_{x\to 0^+} \frac{e^{-1/h}(e^{1/h}-1)}{e^{-1/h}(e^{1/h}+1)}$$

$$= \lim_{x\to 0^+} \frac{1-e^{-1/h}}{1+e^{-1/h}} = \frac{1}{1} = 1 = Rf'(0)$$

$\therefore Lf'(0) \neq Rf'(0)$

f is not derivable at $x=0$

Hence, f is continuous but not derivable at $x=0$.

* Increasing and decreasing functions:

Locally increasing: Let $f: I \to \mathbb{R}$ be a function and $c \in I$. f is said to be locally increasing at c. if $\exists\ \delta > 0 \ni x \in (c-\delta, c) \Rightarrow f(x) < f(c)$ and $x \in (c, c+\delta) \Rightarrow f(x) > f(c)$.

Locally decreasing: Let $f: I \to \mathbb{R}$ be a function and $c \in I$. f is said to be locally decreasing at c if $\delta > 0 \ni x \in (c-\delta, c) \Rightarrow f(x) > f(c)$ and $x \in (c, c+\delta) \Rightarrow f(x) < f(c)$

Theorem: If $f: I \to \mathbb{R}$ is derivable at $c \in I$ and $f'(c) > 0$ then f is locally increasing at c.

proof: Let $f: I \to R$ be derivable at $c \in I$ and $f'(c) > 0$

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c) > 0$$

let $\varepsilon = \frac{f'(c)}{2}$

then $\exists$ $\delta > 0 \ni x \in I$, $|x - c| < \delta$

$$\Rightarrow \left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \varepsilon$$

$$\Rightarrow c - \delta < x < c + \delta$$

$$\Rightarrow f'(c) - \varepsilon < \frac{f(x) - f(c)}{x - c} < f'(c) + \varepsilon$$

$$\Rightarrow c - \delta < x < c + \delta \Rightarrow \frac{f(x) - f(c)}{x - c} > f'(c) - \frac{f'(c)}{2} > 0$$

$$\Rightarrow \frac{f(x) - f(c)}{x - c} > 0 \Rightarrow f(x) - f(c) > 0 \text{ and } x - c > 0$$

$$\text{(or) } f(x) - f(c) < 0 \text{ and } x - c < 0 \quad \text{--- (1)}$$

If $x \in (c - \delta, c)$ then $x < c \Rightarrow x - c < 0$ $\quad$ $x \in (c, c + \delta)$

~~$x <$~~ $\in$ $x > c$

$x - c > 0$

$f(x) - f(c) > 0$

$\Rightarrow f(x) - f(c) < 0$

$\Rightarrow f(x) < f(c)$ [From (1)] $f(x) > f(c)$

$\Rightarrow f(x) > f(c)$

Hence 'f' is locally increasing at 'c'.

Note: We can prove a function $f: I \to R$ is derivable at $c \in I$ and $f'(c) > 0$ then f is locally decreasing at 'c'.

Stationary Point: If $f: I \to R$ is derivable at $c \in I$ and $f'(c)=0$ then f is said to be stationary at c and the point $[c, f(c)]$ is called stationary point.

* Find the interval in which the function $f(x) = x^3 - x - 4$ is increasing and decreasing. Also find the stationary points.

Sol: Given function is $f(x) = x^3 - x - 4$

$$f'(x) = 3x^2 - 1$$

$$f'(x) > 0 \Rightarrow 3x^2 - 1 > 0 \Rightarrow x^2 > \frac{1}{3}$$

$$\Rightarrow x < -\frac{1}{\sqrt{3}} \;\&\; x > \frac{1}{\sqrt{3}}$$

$$\Rightarrow x \in \left(-\infty, -\frac{1}{\sqrt{3}}\right) \cup \left(\frac{1}{\sqrt{3}}, \infty\right)$$

$x^2 - \left(\frac{1}{\sqrt{3}}\right)^2 > 0$

$\left(x - \frac{1}{\sqrt{3}}\right)\left(x + \frac{1}{\sqrt{3}}\right) > 0$

$x - \frac{1}{\sqrt{3}} > 0 \;\&\; x + \frac{1}{\sqrt{3}}$

$x > \frac{1}{\sqrt{3}}$, $x > -\frac{1}{\sqrt{3}}$

$x > \frac{1}{\sqrt{3}}$, $\left(\frac{1}{\sqrt{3}}, \infty\right)$, $\left(-\infty, -\frac{1}{\sqrt{3}}\right)$

$\therefore f$ is increasing in the interval $\left(-\infty, -\frac{1}{\sqrt{3}}\right) \cup \left(\frac{1}{\sqrt{3}}, \infty\right)$

$$f'(x) < 0 \Rightarrow 3x^2 - 1 < 0 \Rightarrow x^2 < \frac{1}{3}, \quad x < \pm\frac{1}{\sqrt{3}}$$

$$\Rightarrow -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}} \Rightarrow x \in \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$, $x < \frac{1}{\sqrt{3}}$, $x > -\frac{1}{\sqrt{3}}$

$\therefore f$ is decreasing in the interval $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$$f'(x) = 0 \Rightarrow x^2 - \frac{1}{3} = 0$$

$$\Rightarrow x = \pm\frac{1}{\sqrt{3}}$$

f is stationary at $+\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$

$$f\left(\frac{1}{\sqrt{3}}\right) = \left(\frac{1}{\sqrt{3}}\right)^3 - \frac{1}{\sqrt{3}} - 4 = \frac{1}{3\sqrt{3}} - \frac{1}{\sqrt{3}} - 4$$

$$= \frac{1-3-12\sqrt{3}}{3\sqrt{3}} = \frac{-2-12\sqrt{3}}{3\sqrt{3}}$$

$$f\left(-\frac{1}{\sqrt{3}}\right) = \left(-\frac{1}{\sqrt{3}}\right)^3 - \left(-\frac{1}{\sqrt{3}}\right) - 4 = \frac{-1}{3\sqrt{3}} + \frac{1}{\sqrt{3}} - 4$$

$$= \frac{-1+3-12\sqrt{3}}{3\sqrt{3}} = \frac{2-12\sqrt{3}}{3\sqrt{3}}$$

∴ The stationary points are

$$\left(\frac{1}{\sqrt{3}}, f\left(\frac{1}{\sqrt{3}}\right)\right) = \left(\frac{1}{\sqrt{3}}, \frac{-2-12\sqrt{3}}{3\sqrt{3}}\right) \text{ and } \left(-\frac{1}{\sqrt{3}}, f\left(-\frac{1}{\sqrt{3}}\right)\right)$$

$$= \left(-\frac{1}{\sqrt{3}}, \frac{2-12\sqrt{3}}{3\sqrt{3}}\right)$$

* Find the interval in which $f(x) = \sqrt{9-x^2}$ is increasing and decreasing. Also find the stationary point?

Sol: Given function is $f(x) = \sqrt{9-x^2}$ $\forall x \in [-3,3]$

[$f(x)$ can be defined in $[-3,3]$ only]

$$f'(x) = \frac{-\cancel{2}x}{\cancel{2}\sqrt{9-x^2}} = \frac{-x}{\sqrt{9-x^2}} \quad \forall x \in (-3,3)$$

$$f'(x) > 0 \Rightarrow \frac{-x}{\sqrt{9-x^2}} > 0 \Rightarrow -x > 0 \Rightarrow x < 0$$

i.e, for $x \in (-3,0)$, $f'(x) > 0$

∴ f is increasing in the interval $(-3,0)$

$$f'(x) < 0 = \frac{-x}{\sqrt{9-x^2}} < 0 \Rightarrow -x < 0 \Rightarrow x > 0$$

i.e, for $x \in (0,3)$, $f'(x) < 0$

$\therefore f$ is decreasing in the interval $(0,3)$

$f'(x)=0 \Rightarrow \dfrac{-x}{\sqrt{9-x^2}}=0 \Rightarrow x=0$

$\therefore f$ is stationary point at 0.

$f(0)=\sqrt{9-0}=\pm 3$

Hence the stationary point $(0,3)$

* Show that ~~$\log(1+x)$~~ $\log(1+x)-\dfrac{2x}{2+x}$ is increasing when $x>0$

Sol: Given let $f(x)=\log(1+x)-\dfrac{2x}{2+x}$

$\dfrac{4+2x-2x}{4+x^2+4x}$

Now $f'(x)=\dfrac{1}{1+x}-\left[\dfrac{(2+x)2-2x(1)}{(2+x)2}\right]$

$=\dfrac{1}{1+x}-\left[\dfrac{4+2x-2x}{4x+4x+x^2}\right]$

$\dfrac{vu'-uv'}{v^2}$

$=\dfrac{4+x^2+4x-4(1+x)}{(1+x)(2+x^2)}$

$\dfrac{4+x^2+4x-4-4x}{(1+x)(2+x)^2}$

$=\dfrac{\cancel{4}+\cancel{4x}+x^2-\cancel{4}-\cancel{4x}}{(1+x)(2+x^2)}=\dfrac{x^2}{(1+x)(2+x^2)}$

$\dfrac{x^2}{(1+x)(2+x)^2}$

when $x>0$

$\therefore f$ is increasing when $x>0$

prove that $\tan x > x > \sin x \ \forall x \in [0, \pi/2]$

consider a function $f(x) = \tan x - x \ \forall x \in [0, \pi/2]$

clearly f is differentiable on the $(0, \pi/2)$

$f'(x) = \sec^2 x - 1 > 0 \ \forall x \in (0, \pi/2)$

$\therefore f$ is increasing on the $(0, \pi/2)$

$\therefore x > 0 \Rightarrow f(x) > f(0)$

$\Rightarrow \tan x - x > 0$

$\Rightarrow \tan x > x$ — (1)

consider a function $g(x) = x - \sin x \ \forall x \in [0, \pi/2]$

clearly G is differentiable on the $(0, \pi/2)$

$G'(x) = 1 - \cos x \ \forall x \in (0, \pi/2)$

$\boxed{\text{cos (or) sin bounded in } (-1, +1)}$

$g'(x) = 1 - \cos x > 0 \ \forall x \in [0, \pi/2]$

$\therefore g$ is increasing on the $[0, \pi/2]$

$\therefore x > 0 \Rightarrow g(x) > g(0)$

$\Rightarrow x - \sin x > 0 - \sin 0$

$\Rightarrow x - \sin x > 0$

$\Rightarrow x > \sin x$ — (2)

from (1) & (2)

$\tan x > x > \sin x \ \forall x \in [0, \pi/2]$

* Show that $\cos x - 1 + \frac{x^2}{2} > 0$ if $x > 0$

Sol: let $f(x) = \cos x - 1 + \frac{x^2}{2}$

clearly f is differentiable on $\mathbb{R}$

$f'(x) = -\sin x - 0 + \frac{2x}{2}$

$f'(x) = x - \sin x > 0 \; \forall \; x > 0$

$\therefore f$ is increasing when $x > 0$

$x > 0 \Rightarrow f(x) > f(0)$

$\Rightarrow \cos x - 1 + \frac{x^2}{2} > \cos 0 - 1 + 0/2$

$\Rightarrow \cos x - 1 + \frac{x^2}{2} > 0$

* Show that $\sin x - x + \frac{x^3}{6} > 0$ if $x > 0$

Sol: let $f(x) = \sin x - x + \frac{x^3}{6}$

$1 - \cos x$
$\nless$
$\sin x > 0$

Clearly f differentiable on $\mathbb{R}$

$f'(x) = \cos x - 1 + \frac{\cancel{3}x^2}{\cancel{6}2}$

$f'(x) = \cos x - 1 + \frac{x^2}{2} > 0 \; \forall \; x > 0$

$[f''(x) = -\sin x - 0 + \frac{\cancel{2}x}{\cancel{2}}]$

$[f'(x) = x - \sin x > 0] \; \forall \; x > 0$

$\therefore f$ is increasing when $x > 0$

$x > 0 \Rightarrow f(x) > f(0)$

$\Rightarrow \sin x - x + \frac{x^3}{6} > \sin 0 - 0 + \frac{0}{6}$

$\Rightarrow \sin x - x + \frac{x^3}{6} > 0$

Scanned with CamScanner

Show that $x < e^x - 1 < \frac{x}{1-x}$ for $x > 0$ and $x \neq 1$

consider a function $f(x) = e^x - 1 - x \;\forall\; x > 0$

clearly f is differentiable $\forall\; x > 0$

$f'(x) = e^x - 0 - 1 \Rightarrow e^x - 1 > 0 \;\forall\; x > 0$

$\therefore$ f is increasing when $x > 0$

$x > 0 \Rightarrow f(x) > f(0)$

$\Rightarrow e^x - 1 - x > e^0 - 1 - 0$

$\Rightarrow e^x - 1 - x > 0$

$\Rightarrow e^x - 1 > x$

$\Rightarrow x < e^x - 1$ — (1)

consider a function $g(x) = e^x - 1 < \frac{x}{1-x} \Rightarrow e^x - 1 - \frac{x}{1-x}$

clearly g is differentiable $\forall\; x > 0$

$g(x) = x - (1-x)(e^x - 1) \;\forall\; x > 0 \;\&\; x \neq 1$

$g'(x) = 1 - (1-x)e^x - (e^x - 1)(-1)$

$= 1 - (1-x)e^x + e^x - 1$

$= \cancel{1} - \cancel{e^x} + xe^x + \cancel{e^x} - \cancel{1}$

$= xe^x > 0$ when $x > 0$

$\therefore$ g is increasing when $x > 0$

$x > 0 \Rightarrow f(x) > f(0)$

$\Rightarrow e^x - 1 - \frac{x}{1-x} > e^0 - 1 - \frac{0}{1-0}$

$\Rightarrow e^x - 1 - \frac{x}{1-x} > 0 \Rightarrow e^x - 1 < \frac{x}{1-x}$

(or) $x > 0 \Rightarrow g(x) > g(0)$

$\Rightarrow x - (1-x)(e^x - 1) > 0 - (1-0)(e^0 - 1)$

$= x - (1-x)(e^x - 1) > 0$

$= x > (1-x)(e^x - 1) \;\forall\; x \neq 1$

$\Rightarrow e^x - 1 < \frac{x}{1-x}$ — (2) $\forall\; x > 0 \;\&\; x \neq 1$

Scanned with CamScanner

from (1) & (2)

$$x < e^x - 1 < \frac{x}{1-x} \quad \forall x > 0 \ \& \ x \neq 1$$

*Theorem:

Darboux's Theorem:- If $f:[a,b] \to \mathbb{R}$ is such that (i) f is derivable on $[a,b]$ and (ii) $f'(a), f'(b)$ have opposite signs (i.e., $f'(a) \cdot f'(b) < 0$) then $\exists\, c \in (a,b) \ni f'(c) = 0$

Proof: Let $f:[a,b] \to \mathbb{R}$ be such that (i) f is derivable on $[a,b]$, (ii) $f'(a) \cdot f'(b) < 0$

Now we have to prove $\exists\, c \in (a,b) \ni f'(c) = 0$

f is derivable on $[a,b] \Rightarrow f$ is continuous on $[a,b]$

$\Rightarrow f$ is bounded and attains its bound on $[a,b]$

Then $\exists\, \alpha, \beta \in [a,b] \ni f(\alpha) = \sup f$, $f(\beta) = \operatorname{Inf} f$ on $[a,b]$

let $f'(a) < 0$ & $f'(b) > 0$

$f'(a) < 0 \Rightarrow f$ is locally decreasing at a.

$\Rightarrow \exists\, \delta_1 > 0 \ni f(x) < f(a) \ \forall x \in [a, a+\delta_1]$ — (1)

If $\alpha - a$ then from (1), $f(x) < f(\alpha) \ \forall x \in [\alpha, \alpha+\delta_1) \subseteq [a,b]$

This is a contraduction

$\therefore \alpha \neq a$

$f'(b) > 0$

$\Rightarrow f$ is locally increasing at b

$\Rightarrow \exists\, \delta_2 > 0 \ni f(x) < f(b) \ \forall x \in (b-\delta_2, b] \subseteq [a,b]$ — (2)

If $\alpha = b$ then $f(x) < f(\alpha) \ \forall x \in (\alpha - \delta_2, \alpha)$

This is a contraduction

$\therefore \alpha \neq b$

Now, we have to prove that $f'(\alpha) < 0$. If $f'(\alpha) < 0$ then is locally decreasing at α $\exists\, \delta_3 > 0 \ni x \in (\alpha-\delta_3, \alpha) \subseteq [a,b]$

$\Rightarrow f(x) < f(\alpha)$

This is a contraduction as $f(\alpha)$ is Inf of f.

$\therefore f'(\alpha) \nless 0$ —(3)

Similary we can prove that $f'(\alpha) \ngtr 0$ —(4)

from (3) & (4) $f'(\alpha) = 0$

similarly, if $f'(a) > 0$ & $f'(b) < 0$ then we can prove $f'(\beta) = 0$

Hence if $f'(a)$ & $f'(b)$ has opposite signs then

$\exists\, c \in (a,b) \ni f'(c) = 0$.

* **Darboux's Intermediate value theorem:**

Statement: If $f:[a,b] \to \mathbb{R}$, f is derivable on $[a,b]$ & $f'(a) \neq f'(b)$, k is a real number between $f'(a)$ & $f'(b)$ then $\exists\, c \in (a,b) \ni f'(c) = k$

proof: Let $f:[a,b] \to \mathbb{R}$ be such that f is derivable on $[a,b]$, $f'(a) \neq f'(b)$ & k is a real number between $f'(a)$ & $f'(b)$ we have to prove $\exists\, c \in (a,b) \ni f'(c) = k$

Define a function $g:[a,b] \to \mathbb{R}$ as $g(x) = f(x) - kx \ \forall x \in [a,b]$

$\because f, x$ are differentiable, g is differentiable on $[a,b]$

$\Rightarrow g'(x) = f'(x) - k$

$f'(a) \neq f'(b)$

$\Rightarrow f'(a) < f'(b)$ (or) $f'(a) > f'(b)$, k is in between $f'(a)$ & $f'(b)$

$\Rightarrow f'(a) < k < f'(b)$ (or) $f'(b) < k < f'(a)$

$\Rightarrow f'(a)-K<0$ & $f'(b)-K>0$ (or) $f'(b)-K<0$ &
$f'(a)-K>0$

$\Rightarrow g'(a)<0$ & $g'(b)>0$ (or) $g'(b)<0$ & $g'(a)>0$

i.e, $g'(a)$ & $g'(b)$ have opposite signs.

Then by Darboux's theorem $\exists\, c \in (a,b) \ni g'(c)=0$

$\Rightarrow f'(c)-K=0 \Rightarrow f'(c)=K$

$\therefore \exists\, c\in(a,b) \ni f'(c)=K$.

* Mean Value Theorem:

Imp ✓

Rolle's theorem: If a function $f:[a,b]\to\mathbb{R}$ is such that (1) f is continuous on $[a,b]$,

(2) f is derivable on (a,b) and

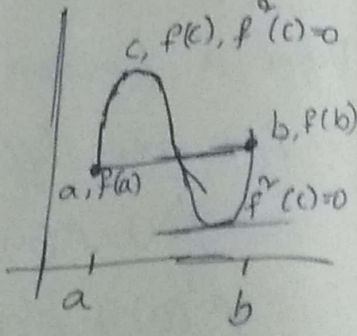
(3) $f(a)=f(b)$, then $\exists\, c\in(a,b) \ni f'(c)=0$

Proof: let $f:[a,b]\to\mathbb{R}$ be such that

(1) f is continuous on $[a,b]$

(2) f is derivable on (a,b) and

(3) $f(a)=f(b)$



Now, we have to prove $\exists\, c\in(a,b) \ni f'(c)=0$

f is continuous on $[a,b] \Rightarrow f$ is bounded and attains its bounds on $[a,b]$

$\Rightarrow \exists\, \alpha,\beta \in [a,b] \ni m = \operatorname{Inf} f$ on $[a,b] = f(\alpha)$ &

$M = \operatorname{Sup} f$ on $[a,b] = f(\beta)$

Scanned with CamScanner

Case (i): let $m = M$

Then $f(x) = \text{Constant} \ \forall \ x \in [a,b]$

$\Rightarrow f'(x) = 0 \ \forall \ x \in (a,b)$

Case (ii): let $m \neq M$

Then either $M \neq f(a)$ so that $M \neq f(b)$

(or)

$m \neq f(a)$ so that $m \neq f(b)$

Suppose that $M \neq f(a)$ and $M \neq f(b)$

$\Rightarrow f(\beta) \neq f(a)$ and $f(\beta) \neq f(b)$

$\Rightarrow \beta \neq a$ and $\beta \neq b$

$\Rightarrow \beta \in (a,b)$

PT

$f'(\beta) = 0$

If possible, suppose that $f'(\beta) < 0$

Then f is locally increasing at β

$\Rightarrow \exists \ \delta > 0 \ni f(x) < f(\beta) \ \forall \ x \in (\beta - \delta, \beta) \subseteq [a,b]$ & — (1)

$f(x) > f(\beta) \ \forall \ x \in (\beta, \beta+\delta) \subseteq [a,b]$ — (2)

(1) is a contraduction as $f(\beta)$ is the sup of f on $[a,b]$

$\therefore f'(\beta) \nless 0$ — (3)

$f'(\beta) \ngtr 0$ — (4)

From (3) & (4) $f'(\beta) = 0$

If $m \neq f(a)$ & $m \neq f(b)$ then

we can prove $f'(\alpha)=0$

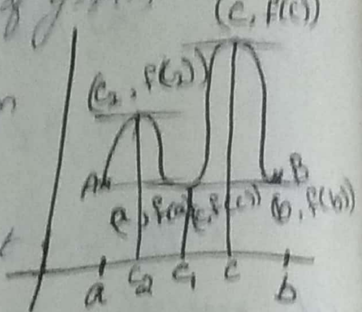
Hence $\exists\ c\in(a,b) \ni f'(c)=0$

Another form of Rolle's Theorem:

Statement: A function $f:[a,b]\to\mathbb{R}$ is such that (i) f is continuous on $[a, a+h]$ (ii) f is derivable on $(a, a+h)$ & (iii) $f(a)=f(a+h)$ then $\exists\ \theta\in(0,1) \ni f'(a+\theta h)=0$

Geometric interpretation of Rolle's theorem:

Statement: Let $f:[a,b]\to\mathbb{R}$ be a function satisfying the 3 conditions of Rolle's theorem. Then the graph of $y=f(x)$ is such that (i) It is a continuous curve from $A(a, f(a))$ to $B(b, f(b))$



(ii) It is a curve having unique tangent line at every intermediate point between A and B.

(iii) The ordinates $f(a)$ & $f(b)$ at the end points A, B are equal

By Rolle's theorem $\exists$ atleast one $c\in(a,b) \ni f'(c)=0$

There is atleast one point $(c, f(c))$ between A and B on the curve at which the tangent line is parallel to x-axis.

* Discuss the applicability of Rolle's Theorem for $f(x)=x^3-6x^2+11x-6$; $a=1$, $b=3$

Sol: Given that $f(x)=x^3-6x^2+11x-6$; $a=1$, $b=3$

since f is a polynomial it is continuous and differentiable on $\mathbb{R}$.

In particular, f is continuous on $[a,b]=[1,3]$ & f is differentiable on $(1,3)$

Now $f(1)=x^3-6x^2+11x-6$

$=(1)^3-6(1)^2+11(1)-6 \Rightarrow 1-6+11-6=0$

$f(3) = (3)^3 - 6(3)^2 + 11(3) - 6$
$= 27 - 54 + 33 - 6 \Rightarrow 60 - 60 = 0$

i.e., $f(1) = f(3)$

f satisfies all conditions of Rolle's theorem

By Rolle's theorem $\exists$ $c \in (1,3) \ni f'(c) = 0$

$f(x) = x^3 - 6x^2 + 11x - 6$

$\Rightarrow f'(x) = 3x^2 - 12x + 11$

$\Rightarrow f'(c) = 3c^2 - 12c + 11 \Rightarrow c = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow \frac{-(-12) \pm \sqrt{144 - 4(3)(11)}}{2 \times 3}$

$\Rightarrow \frac{12 \pm \sqrt{144 - 132}}{6}$

$\Rightarrow \frac{12 \pm 2\sqrt{3}}{6} = \frac{6 \pm \sqrt{3}}{3} = 2 \pm \frac{1}{\sqrt{3}}$

$\therefore$ The values of c for which $f'(c) = 0$ are $2 + \frac{1}{\sqrt{3}} \in (1,3)$ & $2 - \frac{1}{\sqrt{3}} \in (1,3)$

*Examine the applicability of Rolle's theorem $f(x) = 1 - (x-1)^{2/3}$ on $[0,2]$.

Sol. Since f is an algebraic function. It is continuous on $\mathbb{R}$. f is continuous on $[0,2]$. If f is differentiable then $f'(x) = -\frac{2}{3}(x-1)^{\frac{2}{3}-1} \cdot (1)$

$= \frac{-2}{3}(x-1)^{-1/3} = \frac{-2}{3} \frac{1}{(x-1)^{1/3}}$

Since $f'(x)$ does not exist at $x = 1$, f is not differentiable at $1 \in (0,2)$

$\therefore f$ is not differentiable on $(0,2)$

Hence Rolle's theorem is not applicable for f on $[0,2]$

* Show that there is no real number K for which the equation $x^3-3x+K=0$ has two distinct roots on $(0,1)$.

Sol: Suppose that the equ $x^3-3x+K=0$ has two roots α & β in $(0,1)$

Then $f(\alpha)=0$, $f(\beta)=0 \Rightarrow f(\alpha)=f(\beta)$

Since f is a polynomial it is continuous and differentiable on $\mathbb{R}$

In particular f is continuous on $[\alpha,\beta]$ and differentiable on (α,β)

$\therefore f$ is satisfying all conditions of Rolle's theorem.

By Rolle's theorem $\exists\ c\in(\alpha,\beta) \ni f'(c)=0$

Now $f'(x)=3x^2-3$

$f'(c)=0 \Rightarrow 3c^2-3=0 \Rightarrow c^2=3/3 \Rightarrow c=\pm 1 \in (\alpha,\beta) \subseteq (0,1)$

This is a contraduction

$\therefore$ There is no real number K, for which the equation $f(x)=0$ has two roots b/w 0 and 1

✓

* Discuss the applicability of Rolle's theorem for $f(x)=\log\left[\frac{x^2+ab}{x(a+b)}\right]$ in $[a,b]$, $a>0$

Sol: Given function is $f(x)=\log\left[\frac{x^2+ab}{x(a+b)}\right]$ in $[a,b]$, $a>0$

$f(x)=\log(x^2+ab)-\log x(a+b)$

Clearly f is continuous on $[a,b]$ and differentiable on (a,b)

$f(a)=\log\left[\frac{a^2+ab}{a(a+b)}\right]=\log\left[\frac{a^2+ab}{a^2+ab}\right]=\log 1=0$

$f(b)=\log\left[\frac{b^2+ab}{b(a+b)}\right]=\log\left[\frac{b^2+ab}{b^2+ab}\right]=\log 1=0$

i.e, $f(a)=f(b)$

$\therefore f$ satisfies all coditions of Rolle's theorem then by Rolle's theorem $\exists\ c\in(a,b)\ni f'(c)=0$

$f(x)=\log(x^2+ab)-\log x(a+b)$

$f'(x)=\frac{1}{x^2+ab}\cdot 2x-\frac{1}{x(a+b)}\cdot(a+b)$

$f'(x)=\frac{2x}{x^2+ab}-\frac{1}{x(a+b)}(a+b)$

$$f'(x) = \frac{2x}{x^2+ab} - \frac{1}{x}$$

1

$$f'(c) = 0 \Rightarrow \frac{2c}{c^2+ab} - \frac{1}{c} = 0 \Rightarrow \frac{2c^2-(c^2+ab)}{c(c^2+ab)} = 0 \Rightarrow \cancel{\frac{c^2+ab}{c(c^2+ab)}} = 0$$

$$\frac{c^2-ab}{c(c^2+ab)} = 0 \Rightarrow c^2-ab=0 \Rightarrow c^2=ab$$

$$\Rightarrow c = \pm\sqrt{ab} \in (a,b)$$

$\because a>0,\ -\sqrt{ab} \notin (a,b)$ ~~(or) $a>0$ $\sqrt{ab} \notin (a,b)$~~

The value of $c \in (a,b)$ function $f'(c)=0$ is $+\sqrt{ab}$

Show that between any two roots of $e^x \cos x = 1$ there exists atleast one root of $e^x \sin x = 0$

Sol: consider the function $f(x) = \cos x - e^{-x}$

clearly f is continuous and differentiable on $\mathbb{R}$.

Let α, β be the two roots of $f(x)=0$

i.e., $f(\alpha)=0,\ f(\beta)=0$

$f(x)=0$ can be written as $\cos x = e^{-x} \Rightarrow e^x \cos x = 1$

i.e., α, β are two roots of $e^x \cos x = 1$

clearly, f is continuous on $[\alpha, \beta]$

f is differentiable on (α, β) &

we have $f(\alpha) = f(\beta)$

$\therefore f$ satisfies all conditions of rolle's theorem in $[\alpha, \beta]$

By Rolle's theorem $\exists\ c \in (\alpha, \beta) \ni f'(c)=0$

$$f'(x) = -\sin x + e^{-x}$$

$$f'(c) \Rightarrow -\sin c + e^{-c} = 0$$

$$\Rightarrow \sin c = e^{-c}$$

$$\Rightarrow e^c \cdot \sin c = 1$$

Scanned with CamScanner

Imp

Lagrange's mean value theorem: (or) First mean value theorem

Statement: If $f:[a,b]\rightarrow\mathbb{R}$ is such that

(4)

(i) f is continuous on $[a,b]$

(ii) f is derivable on (a,b) then $\exists\ c\in(a,b) \ni f'(c)=\frac{f(b)-f(a)}{b-a}$

Proof: let $f:[a,b]\rightarrow\mathbb{R}$ be such that

(i) f is continuous on $[a,b]$

(ii) f is derivable on (a,b)

Define a function $\phi:[a,b]\rightarrow\mathbb{R}$ as

$\phi(x)=f(x)+kx$, where k is a real number choosen

such that $\phi(a)=\phi(b)$

$\phi(a)=\phi(b) \Rightarrow f(a)+ka=f(b)+kb$

$\Rightarrow k(a-b)=f(b)-f(a)$

$$\Rightarrow -k=\frac{f(b)-f(a)}{b-a} \quad \text{---(1)}$$

$\because f$ is continuous on $[a,b]$ & k is continuous on $\mathbb{R}$,

ϕ is continuous on $[a,b]$

$\because f$ is derivable on (a,b) & k is derivable on $\mathbb{R}$,

ϕ is derivable on (a,b)

$\therefore \phi$ satisfies all conditions of Rolle's theorem.

$\therefore$ By Rolle's theorem $\exists\ c\in(a,b) \ni \phi'(c)=0 \Rightarrow f'(c)+k=0$

$\Rightarrow -k=f'(c)$ ---(2)

from (1) & (2)

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Another form of lagrange's theorem: If f a function $f:[a, a+h] \to \mathbb{R}$ is such that

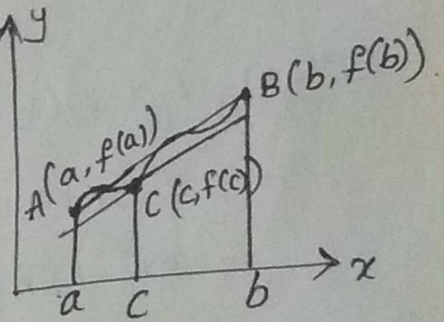
(i) f is continuous on $[a, a+h]$

(ii) f is derivable on $(a, a+h)$ then $\exists\ \theta \in (0,1) \ni$ ~~$f'(a+\theta h)$~~

$$f'(a+\theta h) = \frac{f(a+h)-f(a)}{h}$$

Geometrical Interpretation of Lagrange's theorem:

Let $f:[a,b] \to \mathbb{R}$ be a function satisfying the ~~lag~~ two conditions of lagrange's theorem.



Then the graph $y = f(x)$ is such that

(i) It is a continuous curve from the point $A(a, f(a))$ to the point $B(b, f(b))$

(ii) It is a curve having unique tangent line at every intermediate point between A and B.

$\frac{f(b)-f(a)}{b-a}$ is slope of line joining A and B

$f'(c)$ is slope of the tangent line at $C(c, f(c))$.

$\therefore \exists\ c \in (a,b) \ni f'(c) = \frac{f(b)-f(a)}{b-a} \Rightarrow$ There exists a point $C(c, f(c))$ in between A and B such that the tangent lin at c is parallel to the line Joining A and B

* Find c of lagrange's mean value theorem $f(x)=(x-1)(x-2)(x-3)$ on $[0,4]$

Sol: Given function is $f(x)=(x-1)(x-2)(x-3)$ on $[0,4]$

Since f is a polynomial it is continuous and differentiable on $\mathbb{R}$

In particular, f is continuous on $[0,4]$ & derivable on $(0,4)$

$\therefore$ f satisfies all conditions of lagrange's theorem.

By lagrange's theorem $\exists\, c \in (0,4) \ni f'(c) = \dfrac{f(4)-f(0)}{4-0}$

$f'(x) = 1(x-2)(x-3) + 1(x-1)(x-3) + 1(x-1)(x-2)$

$= x^2-5x+6+x^2-4x+3+x^2-3x+2$

$= 3x^2-12x+11$

$f'(c) = 3c^2-12c+11$

$f(4) = (4-1)(4-2)(4-3)$

$= (3)(2)(1) = 6$

$f(0) = (0-1)(0-2)(0-3)$

$= (-1)(-2)(-3) = -6$

$f'(c) = \dfrac{f(4)-f(0)}{4-0}$

$\Rightarrow 3c^2-12c+11 = \dfrac{6+6}{4} = \dfrac{12}{4}$

$\Rightarrow 3c^2-12c+11 = 3 \qquad \Rightarrow 3c^2-12c+8=0$

$$c = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{12 \pm \sqrt{144-96}}{6}$$

$$= \frac{12 \pm \sqrt{48}}{6} = \frac{12 \pm 4\sqrt{3}}{6} = 2 \pm \frac{2}{3}\sqrt{3}$$

$$= 2 \pm \frac{2}{\sqrt{3}} \in (0,4)$$

From lagrang

$\therefore$ The values of c from lagranges theorem for the given function is $2+\frac{2}{\sqrt{3}}$ & $2-\frac{2}{\sqrt{3}}$

* Examine the applicability of Rolle's theorem $f(x) = 1-(x-1)^{2/3}$ on $[0,2]$

Sol: Since f is an alzebric function, it is continuous on $\mathbb{R}$.

f is continuous on $[0,2]$

If f is differentiable then $f'(x) = -\frac{2}{3}(x-1)^{\frac{2}{3}-1} \cdot (1)$

$$= -\frac{2}{3}(x-1)^{-1/3} = \frac{-2}{3} \cdot \frac{1}{(x-1)^{1/3}}$$

Since $f'(x)$ does not exist at $x=1$, f is not differentiable at $1 \in (0,2)$

$\therefore f$ is not differentiable on $(0,2)$

Hence Rolle's theorem is not applicable for f on $[0,2]$

* Find the θ of lagrange's theorem for $f(x) = x^2-2x+3$; $a=1, h=\frac{1}{2}$

Sol: Given function is $f(x) = x^2-2x+3$

Given that $a=1$, $h=\frac{1}{2}$

Since f is a polynomial it is continuous and differentiable on $\mathbb{R}$

In partiation f is continuous on $[a,a+h] = [1, 1+\frac{1}{2}] = [1, \frac{3}{2}]$

f is differentiable on $(a, a+h) = (1, \frac{3}{2})$

$\therefore f$ satisfies all conditions for lagrange's theorem.

By lagrange's theorem, $\exists\ \theta \in (0,1) \ni f'(a+\theta h) = \dfrac{f(a+h)-f(a)}{h}$

$$f(a+h) = f\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^2 - 2\left(\frac{3}{2}\right) + 3$$

$$= \frac{9}{4} - 3 + 3 = \frac{9}{4}$$

$$f(a) = f(1) = 1^2 - 2(1) + 3 = 2$$

$f(x) = x^2 - 2x + 3;$

$f'(x) = 2x - 2$

$$f'(a+\theta h) = f'\left(1+\frac{\theta}{2}\right) = 2\left(1+\frac{\theta}{2}\right) - 2$$

$$= 2 + \theta - 2 = \theta$$

Now, $f'(a+\theta h) = \dfrac{f(a+h) - f(a)}{h}.$

$$\Rightarrow \theta = \frac{\frac{9}{4} - 2}{1/2} = \frac{1}{4} \times \frac{2}{1} = \frac{1}{2} \in (0,1)$$

* Discuss the applicability of lagrange's ~~said~~ mean value theorem for $f(x) = x(x-1)(x-2)$ on $\left[0, \frac{1}{2}\right]$

Sol: Given function is $f(x) = x(x-1)(x-2)$

Since f is a polynomial it is continuous and ~~differen~~ differentiable on $\mathbb{R}$.

In particular f is continuous on $\left[0, \frac{1}{2}\right]$ & derivable on $\left(0, \frac{1}{2}\right)$

$\therefore f$ satisfies all conditions of lagrange's theorem

$\therefore$ By lagrange's theorem $\exists\, c \in (0, \frac{1}{2}) \ni f'(c) = \dfrac{f(\frac{1}{2}) - f(0)}{\frac{1}{2} - 0}$

$$f'(x) = 1(x-1)(x-2) + x(1)(x-2) + x(x-1)(1)$$
$$= (x-1)(x-2) + x(x-2) + x(x-1)$$
$$= x^2 - 3x + 2 + x^2 - 2x + x^2 - x$$
$$= 3x^2 - 6x + 2$$

$$f'(c) = 3c^2 - 6c + 2$$

$$f(\tfrac{1}{2}) = \tfrac{1}{2}(\tfrac{1}{2} - 1)(\tfrac{1}{2} - 2)$$
$$= \tfrac{1}{2}(-\tfrac{1}{2})(-\tfrac{3}{2}) = \frac{3}{8}$$

$$f(0) = 0$$

$$\therefore f'(c) = \frac{f(\frac{1}{2}) - f(0)}{\frac{1}{2} - 0}$$

$$\Rightarrow 3c^2 - 6c + 2 = \frac{\frac{3}{8} - 0}{\frac{1}{2}} \Rightarrow 3c^2 - 6c + 2 = \frac{3}{\cancel{8}_4} \times \frac{\cancel{2}}{1}$$

$$\Rightarrow 3c^2 - 6c + 2 = \frac{3}{4}$$

$$\Rightarrow 12c^2 - 24c + 8 - 3 = 0$$

$$\Rightarrow 12c^2 - 24c + 5 = 0$$

$$c = \frac{-(-24) \pm \sqrt{(24)^2 - 4(12)(5)}}{2(12)}$$

$$= \frac{24 \pm \sqrt{576 - 240}}{24}$$

$$= \frac{24 \pm \sqrt{336}}{24}$$

$$= \frac{24 \pm 4\sqrt{21}}{24} = \frac{6 \pm \sqrt{21}}{6}$$

$$= 1 \pm \frac{\sqrt{21}}{6}$$

$\because 1+\frac{\sqrt{21}}{6} > 1$ & $c \in (0, \frac{1}{2})$, $c \neq 1+\frac{\sqrt{21}}{6}$

$\therefore$ The value of c is $1-\frac{\sqrt{21}}{6}$

*Show that $\frac{v-u}{1+v^2} < \tan^{-1}v - \tan^{-1}u < \frac{v-u}{1+u^2}$ for $0<u<v$.
Hence deduce that $\frac{\pi}{4}+\frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4}+\frac{1}{6}$.

Sol: let $f(x) = \tan^{-1}x \ \forall x>0$

For $0<u<v$, f is continuous on $[u,v]$ & differentiable on (u,v)

f satisfies all conditions of lagrange's theorem on $[u,v]$

By lagrange's theorem, $\exists\ c \in (u,v) \ni f'(c) = \frac{f(v)-f(u)}{v-u}$

$$\frac{1}{1+c^2} = \frac{\tan^{-1}v - \tan^{-1}u}{v-u} \quad \text{---(1)}$$

$c \in (u,v) \Rightarrow u<c<v$

$\Rightarrow u^2<c^2<v^2$

$\Rightarrow 1+u^2<1+c^2<1+v^2$

$\Rightarrow \frac{1}{1+u^2} > \frac{1}{1+c^2} > \frac{1}{1+v^2}$

$\Rightarrow \frac{1}{1+v^2} < \frac{1}{1+c^2} < \frac{1}{1+u^2}$

$\Rightarrow \frac{1}{1+v^2} < \frac{\tan^{-1}v - \tan^{-1}u}{v-u} < \frac{1}{1+u^2}$ [From (1)]

$$\Rightarrow \frac{v-u}{1+v^2} < \tan^{-1}v - \tan^{-1}u < \frac{v-u}{1+u^2} \quad \text{---(2)}$$

Put $u=1$ & $v=\frac{4}{3}$ in (2)

$$\frac{\frac{4}{3}-1}{1+\frac{16}{9}} < \tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}(1) < \frac{\frac{4}{3}-1}{1+1}$$

Scanned with CamScanner

$\Rightarrow \frac{1}{3} \times \frac{\cancel{9}\,3}{25} < \tan^{-1}\frac{4}{3} - \frac{\pi}{4} < \frac{\cancel{1}}{3} \times \frac{\cancel{2}\,1}{\cancel{4}\,2}$

$\cancel{\Rightarrow \frac{3}{25} < \tan^{-1}\frac{4}{3} - \frac{\pi}{4} < \frac{1}{6}}$ $\Rightarrow \frac{3}{25} < \tan^{-1}\frac{4}{3} + \frac{\pi}{4} < \frac{1}{6}$

$\Rightarrow \frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$

* Using lagrange's theorem show that $x > \log(1+x) > \frac{x}{1+x}$ if $f(x) = \log(1+x)\ \forall\ x > 0$

Sol: Consider a function $f(x) = \log(1+x)\ \forall\ x \in [0,t], t > 0$

clearly f is continuous on $[0,t]$ & derivable on $(0,t)$

Now, by lagrange's theorem $\exists\ (0,t) \ni f'(c) = \frac{f(t)-f(0)}{t-0}$

$$\Rightarrow \frac{1}{1+c} = \frac{\log(1+t) - \log(1)}{t}$$

$$\Rightarrow \frac{1}{1+c} = \frac{\log(1+t)}{t} \quad \text{---(1)} \quad [\because \log 1 = 0]$$

$c \in (0,t) \Rightarrow 0 < c < t$

$$\Rightarrow 1 < c+1 < 1+t$$

$$\Rightarrow 1 > \frac{1}{1+c} > \frac{1}{1+t}$$

$$\Rightarrow 1 > \frac{\log(1+t)}{t} > \frac{1}{1+t} \quad \text{(From (1))}$$

$$\Rightarrow t > \log(1+t) > \frac{t}{1+t}$$

$$\therefore x > \log(1+x) > \frac{x}{1+x}\ \forall\ x > 0$$

Scanned with CamScanner

* Show that $\frac{\pi}{6}+\frac{\sqrt{3}}{15}<\sin^{-1}(0.6)<\frac{\pi}{6}+\frac{1}{8}$

Sol: Define a function $f(x)=\sin^{-1}x \ \forall \ x\in[a,b]$

where $a>0$ & $b<1$

Clearly f is continuous and differentiable on $[a,b]$

Now by lagrange's theorem $\exists\ c\in(a,b)\ \ni f'(c)=\frac{f(b)-f(a)}{b-a}$

$$\Rightarrow \frac{1}{\sqrt{1-c^2}}=\frac{\sin^{-1}b-\sin^{-1}a}{b-a} \quad \text{---}(1)$$

$c\in(a,b) \Rightarrow a<c<b$

$\Rightarrow a^2<c^2<b^2$

$\Rightarrow -a^2>-c^2>-b^2$

$\Rightarrow 1-a^2>1-c^2>1-b^2$

$\Rightarrow \sqrt{1-a^2}>\sqrt{1-c^2}>\sqrt{1-b^2}$

$\Rightarrow \frac{1}{\sqrt{1-a^2}}<\frac{1}{\sqrt{1-c^2}}<\frac{1}{\sqrt{1-b^2}}$

$$\Rightarrow \frac{1}{\sqrt{1-a^2}}<\frac{\sin^{-1}b-\sin^{-1}a}{b-a}<\frac{1}{\sqrt{1-b^2}} \quad [\text{From}(1)]$$

$$\Rightarrow \frac{b-a}{\sqrt{1-a^2}}<\sin^{-1}b-\sin^{-1}a<\frac{b-a}{\sqrt{1-b^2}}$$

Put $a=\frac{1}{2}=0.5$, $b=0.6=\frac{6}{10}=\frac{3}{5}$

$$\Rightarrow \frac{0.6-0.5}{\sqrt{1-(0.5)^2}} < \sin^{-1}(0.6) - \sin^{-1}(0.5) < \frac{0.6-0.5}{\sqrt{1-(0.6)^2}}$$

$$\Rightarrow \frac{0.1}{\sqrt{0.75}} < \sin^{-1}(0.6) - \frac{\pi}{6} < \frac{0.1}{\sqrt{0.64}}$$

$$\Rightarrow \frac{0.1}{\sqrt{\frac{75}{100}}} + \frac{\pi}{6} < \sin^{-1}(0.6) < \frac{0.1}{\sqrt{64/100}} + \frac{\pi}{6}$$

$$\Rightarrow \frac{0.1}{5\sqrt{3}} + \frac{\pi}{6} < \sin^{-1}(0.6) < \frac{0.1 \times 10}{8} + \frac{\pi}{6}$$

$$\Rightarrow \frac{\sqrt{3}}{15} + \frac{\pi}{6} < \sin^{-1}(0.6) < \frac{1}{8} + \frac{\pi}{6}.$$

* Prove that $10.22 < \sqrt{105} < 10.25$, $f(x) = \sqrt{x}$ $\forall x \in [100, 105]$

Sol: cosider a function $f(x) = \sqrt{x}$ $\forall x \in [100, 105]$

clearly f is continuous on $[100, 105]$ & derivable on $(100, 105)$

Now by lagrange's theorem $\exists$ $c \in (100, 105) \ni$

$$f'(c) = \frac{f(105) - f(100)}{105 - 100}$$

$$\Rightarrow \frac{1}{2\sqrt{c}} = \frac{\sqrt{105} - 10}{5} \longrightarrow (1)$$

$c \in (100, 105) \Rightarrow 100 < c < 105$

$$\Rightarrow \sqrt{100} < \sqrt{c} < \sqrt{105}$$

$$\Rightarrow 2\sqrt{100} < 2\sqrt{c} < 2\sqrt{105}$$

$$\Rightarrow \frac{1}{2\sqrt{100}} < \frac{1}{2\sqrt{c}} < \frac{1}{2\sqrt{105}}$$

$\Rightarrow \frac{1}{2\sqrt{100}} > \frac{\sqrt{105}-10}{5} > \frac{1}{2\sqrt{105}}$ [From (1)]

$\Rightarrow \frac{1}{2\sqrt{105}} < \frac{\sqrt{105}-10}{5} < \frac{1}{2\sqrt{100}}$

$\Rightarrow \frac{5}{2\sqrt{105}} < \sqrt{105}-10 < \frac{5}{2\sqrt{100}}$

$\Rightarrow \frac{5}{20.49} < \sqrt{105}-10 < \frac{5}{2\sqrt{100}}$

$\Rightarrow 0.24 < \sqrt{105}-10 < 0.25$

$\Rightarrow 10+0.24 < \sqrt{105} < 10+0.25$

$\Rightarrow 10.24 < \sqrt{105} < 10.25.$

Imp

Cauchy's Mean value theorem:

Statement: Let $f: [a,b] \to \mathbb{R}$ & $g: [a,b] \to \mathbb{R}$ be two functions such that

(i) f, g are continuous on $[a,b]$

(ii) f, g are derivable on (a,b) and

(iii) $g'(x) \neq 0 \ \forall x \in (a,b)$ then $\exists\, c \in (a,b) \ni$

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Proof: let $f: [a,b] \to \mathbb{R}$ & $g: [a,b] \to \mathbb{R}$ be two functions such that

(i) f, g are continuous on $[a,b]$

(ii) f, g are derivable on (a,b)

(iii) $g'(x) \neq 0 \ \forall \ x \in (a,b)$ ~~the~~

Define a function $\phi : [a,b] \rightarrow \mathbb{R}$ as $\phi(x) = f(x) + Kg(x)$,

where K is a real number choosen

$\Rightarrow \phi(a) = \phi(b)$

$\phi(a) = \phi(b) \Rightarrow f(a) + Kg(a) = f(b) + Kg(b)$

$\Rightarrow K[g(a) - g(b)] = f(b) - f(a)$ —— (1)

If $g(a) - g(b) = 0$ i.e., $g(a) = g(b)$

Then g satisfies all conditions of Rolle's theorem

$\therefore \exists \ c \in (a,b) \ni g'(c) = 0$

This is a contraduction as $g'(x) \neq 0 \ \forall \ x \in (a,b)$

$\therefore g(a) - g(b) \neq 0$

$\therefore$ equ (1) $\Rightarrow K = \dfrac{f(b) - f(a)}{g(a) - g(b)}$

$\Rightarrow -K = \dfrac{f(b) - f(a)}{g(b) - g(a)}$ —— (2)

$\because f$ & g are continuous on $[a,b]$, ϕ is continuous on $[a,b]$

$\because f, g$ are derivable on (a,b), ϕ is derivable on (a,b)

$\therefore \phi$ satisfies all conditions of Rolle's theorem.

$\therefore$ By rolles theorem $\exists\, c \in (a,b) \ni \phi'(c) = 0$

$$\Rightarrow f'(c) + K g'(c) = 0$$

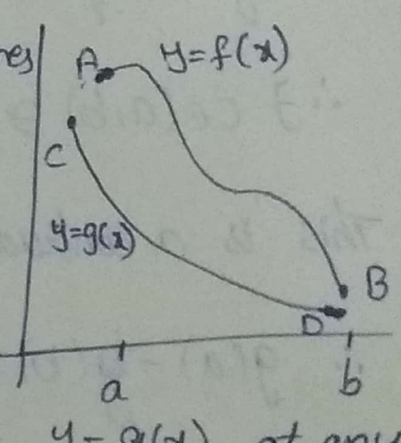
$$\Rightarrow -K = \frac{f'(c)}{g'(c)} \quad \text{--- (3)}$$

From (2) & (3) $$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

* Geometrical Interpretation of cauchy's theorem:

(1) $y=f(x)$ and $y=g(x)$ are two continuous curves between the points $A(a, f(a))$, $B(b, f(b))$ & $C(c, g(a))$, $D(d, g(d))$ respectively.

(2) These curves have unique tangent lines ~~at~~ between any intermediate points A, B & C, D respectively.



(3) Slope of the tangent line to the curve $y=g(x)$ at any point between C & D is non zero.

Then $\exists$ a ~~point~~ value c in between a & b $\ni$ the ratio ~~slopes~~ of the slopes of the tangent lines of the curves $y=f(x)$ & $y=g(x)$ at $(c, f(c))$, $(c, g(c))$ is same as the ratio of the slopes of the chords $\overline{AB}$ & $\overline{CD}$

Another form of Cauchy's mean value theorem:

Statement: If $f: [a, a+h] \to \mathbb{R}$ such that

(i) f, g are continuous on $[a, a+h]$

(ii) f, g are derivable on $(a, a+h)$

(iii) $g'(x) \neq 0 \ \forall x \in (a, a+h)$ then $\exists \ \theta \in (0,1)$

such that $\dfrac{f(a+h) - f(a)}{g(a+h) - g(a)} = \dfrac{f'(a+\theta h)}{g'(a+\theta h)}$

* Find the value of c by using cauchy mean value theorem for $f(x) = \dfrac{1}{x^2}$ & $g(x) = \dfrac{1}{x}$ on $[a,b]$ where $a, b > 0$

Sol: Given $f(x) = \dfrac{1}{x^2}$ & $g(x) = \dfrac{1}{x}$

$\Rightarrow f'(x) = \dfrac{-2}{x^3}, \ g(x) = \dfrac{-1}{x^2} \neq 0$ in (a,b) where $a, b > 0$

Clearly f, g are continuous on $[a,b]$ &

f, g are derivable on (a,b)

$\therefore f, g$ satisfies all the conditions of cauchy mean value theorem

Then $\exists \ c \in (a,b) \ni \dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$

$$\Rightarrow \frac{-2/c^3}{-1/c^2} = \frac{1/b^2 - 1/a^2}{1/b - 1/a}$$

$$\Rightarrow \frac{2}{c^3} \times \frac{c^2}{1} = \frac{a^2-b^2}{a^2b^2} \times \frac{a+b}{a-b}$$

$$\Rightarrow \frac{2}{c} = \frac{a+b}{ab}$$

$$\Rightarrow \frac{c}{2} = \frac{ab}{a+b}$$

$$\Rightarrow c = \frac{2ab}{a+b} \in (a,b)$$

* Find the value of c by using cauchy mean value theorem for $f(x)=\sqrt{x}$ & $g(x)=\frac{1}{\sqrt{x}}$ in $[a,b]$ where $0<a<b$

Sol: Given $f(x)=\sqrt{x}$ & $g(x)=\frac{1}{\sqrt{x}}$

Since $x^n, n\in\mathbb{R}$, x^n is continuous on $\mathbb{R} \Rightarrow x^n$ is continuous on $[a,b]$

$f'(x)=\frac{1}{2\sqrt{x}}$; $g'(x)=\frac{-1}{2x\sqrt{x}}$ exists and $g'(x)\neq 0$ in (a,b) where $0<a<b$

clearly f,g are continuous on $[a,b]$ &

f,g are derivable on (a,b)

$\therefore f,g$ satisfies all conditions of cauchys mean value theorem.

Then $\exists\ c\in(a,b) \ni \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

$$\Rightarrow \frac{\frac{1}{2\sqrt{x}}}{-1/2x\sqrt{x}} = \frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}}$$

$$\Rightarrow -c = \frac{\sqrt{b}-\sqrt{a}}{\frac{\sqrt{a}-\sqrt{b}}{\sqrt{ab}}}$$

$$\Rightarrow \cancel{-}c = \frac{\cancel{-}(\cancel{\sqrt{a}-\sqrt{b}})}{\cancel{\sqrt{a}-\sqrt{b}}} \times \sqrt{ab}$$

$$\Rightarrow c = \sqrt{ab}$$

* verify cauchy mean value theorem for $f(x)=x^2$ & $g(x)=x^3$ in $[1,2]$

Sol: $f(x)=x^2$, $g(x)=x^3$

$f'(x)=2x$, $g'(x)=3x^2$ exists and $g'(x)\neq 0$ in $(1,2)$

clearly ~~g~~ f, g are continuous on $[a,b]$ &

f, g are derivable on (a,b)

$\therefore f, g$ satisfies all conditions of cauchys mean value theorem

Then $\exists$ ~~c~~ $c\in(1,2) \ni \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

$$\Rightarrow \frac{2c}{3c^2} = \frac{(2)^2-(1)^2}{(2)^3-(1)^3}$$

$$\Rightarrow \frac{2}{3c} = \frac{4-1}{8-1}$$

$\Rightarrow \frac{2}{3c} = \frac{3}{7}$

$\Rightarrow \frac{3c}{2} = \frac{7}{3}$

$\Rightarrow c = \frac{14}{9} \in (1,2)$

* Find the value of c by using cauchy mean value theorem for the function $f(x) = e^x$, $g(x) = e^{-x}$ on $[a,b]$ where $a,b>0$

Sol: Given $f(x) = e^x$, $g(x) = e^{-x}$

$f'(x) = e^x$, $g'(x) = -e^{-x}$ exists and $g'(x) \neq 0$ in (a,b) where $a,b>0$

clearly f,g are continuous on $[a,b]$ &

f,g are derivable on (a,b)

$\therefore f,g$ satisfies all the conditions of cauchy mean value theorem.

Then $\exists\; c \in (a,b) \ni \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

$\Rightarrow \frac{e^c}{-e^{-c}} = \frac{e^b - e^a}{e^{-b} - e^{-a}}$

$\Rightarrow -e^c \times e^c = \frac{e^b - e^a}{\frac{1}{e^b} - \frac{1}{e^a}}$

$\Rightarrow e^{-2c} = \frac{e^b - e^a}{e^a - e^b} \times e^a e^b$

$$\Rightarrow -e^{2x} = \frac{-(\cancel{e^a - e^b})}{\cancel{e^a - e^b}} e^a e^b$$

$$\Rightarrow e^{2c} = e^{a+b}$$

$$\Rightarrow 2c = a+b$$

$$\Rightarrow c = \frac{a+b}{2} \in (a,b)$$